EXERCISE :- 8.1

Question 1

Expand the expression $(1-2x)^5$

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as

$$\begin{aligned}&(1-2x)^5\\&={}^5C_0(1)^5-{}^5C_1(1)^4(2x)+{}^5C_2(1)^3(2x)^2-{}^5C_3(1)^2(2x)^3+{}^5C_4(1)^1(2x)^4-{}^5C_5(2x)^5\\&=1-5(2x)+10(4x^2)-10(8x^3)+5(16x^4)-(32x^5)\\&=1-10x+40x^2-80x^3+80x^4-32x^5\end{aligned}$$

Question 2:

Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$\begin{aligned}\left(\frac{2}{x}-\frac{x}{2}\right)^5&={}^5C_0\left(\frac{2}{x}\right)^5-{}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right)+{}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2\\&\quad-{}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3+{}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4-{}^5C_5\left(\frac{x}{2}\right)^5\\&=\frac{32}{x^5}-5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right)+10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right)-10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right)+5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right)-\frac{x^5}{32}\\&=\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^5}{32}\end{aligned}$$

Question 3:

Expand the expression $(2x-3)^6$

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$$(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3$$
$$+ {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6$$
$$= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27)$$
$$+15(4x^2)(81) - 6(2x)(243) + 729$$
$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

Question 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2$$
$$+ {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$
$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$
$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

Question 5:

Expand $\left(x+\frac{1}{x}\right)^6$

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$
$$+ {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$
$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$
$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 96 = 100 – 4

$$\therefore (96)^3 = (100-4)^3$$
$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$$
$$= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$$
$$= 1000000 - 120000 + 4800 - 64$$
$$= 884736$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 102 = 100 + 2

$$\begin{aligned}\therefore (102)^5 &= (100+2)^5\\ &= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3\\ &\quad + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5\\ &= (100)^5 + 5(100)^4(2) + 10(100)^3(2)^2 + 10(100)^2(2)^3 + 5(100)(2)^4 + (2)^5\\ &= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32\\ &= 11040808032\end{aligned}$$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 101 = 100 + 1

$$\begin{aligned}\therefore (101)^4 &= (100+1)^4\\ &= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4\\ &= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4\\ &= 100000000 + 4000000 + 60000 + 400 + 1\\ &= 104060401\end{aligned}$$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 99 = 100 – 1

$$\therefore (99)^5 = (100-1)^5$$
$$= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$$
$$= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1$$
$$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$$
$$= 10010000500 - 500100001$$
$$= 9509900499$$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$$(1.1)^{10000} = (1+0.1)^{10000}$$
$$= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms}$$
$$= 1 + 10000 \times 1.1 + \text{Other positive terms}$$
$$= 1 + 11000 + \text{Other positive terms}$$
$$> 1000$$

Hence, $(1.1)^{10000} > 1000$

Question 11:

Find $(a+b)^4 - (a-b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

Using Binomial Theorem, the expressions, $(a+b)^4$ and $(a-b)^4$, can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$
$$\begin{aligned}\therefore (a+b)^4-(a-b)^4 &= {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 \\ &\quad -\left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right] \\ &= 2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b+4ab^3\right) \\ &= 8ab\left(a^2+b^2\right)\end{aligned}$$

By putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain

$$\begin{aligned}\left(\sqrt{3}+\sqrt{2}\right)^4-\left(\sqrt{3}-\sqrt{2}\right)^4 &= 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2+\left(\sqrt{2}\right)^2\right\} \\ &= 8\left(\sqrt{6}\right)\{3+2\} = 40\sqrt{6}\end{aligned}$$

Question 12:

Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$.

Using Binomial Theorem, the expressions, $(x + 1)^6$ and $(x - 1)^6$, can be expanded as

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$
$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$
$$\begin{aligned}\therefore (x+1)^6+(x-1)^6 &= 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right] \\ &= 2\left[x^6+15x^4+15x^2+1\right]\end{aligned}$$

By putting $x=\sqrt{2}$, we obtain

$$\begin{aligned}\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6 &= 2\left[\left(\sqrt{2}\right)^6+15\left(\sqrt{2}\right)^4+15\left(\sqrt{2}\right)^2+1\right] \\ &= 2\left(8+15\times4+15\times2+1\right) \\ &= 2\left(8+60+30+1\right) \\ &= 2(99) = 198\end{aligned}$$

Question 13:

Show that $9^{n+1}-8n-9$ is divisible by 64, whenever n is a positive integer.

In order to show that $9^{n+1}-8n-9$ is divisible by 64, it has to be proved that,

$9^{n+1}-8n-9=64k$, where *k* is some natural number

By Binomial Theorem,

$$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + \ldots + {}^mC_ma^m$$

For *a* = 8 and *m* = *n* + 1, we obtain

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$$
$$\Rightarrow 9^{n+1} = 1+(n+1)(8)+8^2\left[{}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$
$$\Rightarrow 9^{n+1} = 9+8n+64\left[{}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$
$$\Rightarrow 9^{n+1}-8n-9 = 64k,\ \text{where } k = {}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\ \text{ is a natural number}$$

Thus, $9^{n+1}-8n-9$ is divisible by 64, whenever *n* is a positive integer.

Question 14:

Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

By Binomial Theorem,

$$\sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r = (a+b)^n$$

By putting *b* = 3 and *a* = 1 in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r (1)^{n-r}(3)^r = (1+3)^n$$
$$\Rightarrow \sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$$

Hence, proved.

EXERCISE :- 8.2

Question 1:

Find the coefficient of x^5 in $(x + 3)^8$

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^5 occurs in the $(r + 1)^{th}$ term of the expansion $(x + 3)^8$, we obtain

$$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 and in T_{r+1}, we obtain

$r = 3$

Thus, the coefficient of x^5 is $${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2.5!} \cdot 3^3 = 1512$$

Question 2:

Find the coefficient of a^5b^7 in $(a - 2b)^{12}$

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that a^5b^7 occurs in the $(r + 1)^{th}$ term of the expansion $(a - 2b)^{12}$, we obtain

$$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$$

Comparing the indices of a and b in a^5 b^7 and in T_{r+1}, we obtain

$r = 7$

Thus, the coefficient of a^5b^7 is
$${}^{12}C_7 (-2)^7 = -\frac{12!}{7!5!} \cdot 2^7 = -\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8.7!}{5 \cdot 4 \cdot 3 \cdot 2.7!} \cdot 2^7 = -(792)(128) = -101376$$

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$$T_{r+1} = {}^6C_r \left(x^2\right)^{6-r} \left(-y\right)^r = \left(-1\right)^r \, {}^6C_r . x^{12-2r} . y^r$$

Question 4:

Write the general term in the expansion of $(x^2 - yx)^{12}$, $x \neq 0$

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of$(x^2 - yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} \left(-yx\right)^r = \left(-1\right)^r \, {}^{12}C_r . x^{24-2r} . y^r . x^r = \left(-1\right)^r \, {}^{12}C_r . x^{24-r} . y^r$$

Question 5:

Find the 4^{th} term in the expansion of $(x - 2y)^{12}$.

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the 4^{th} term in the expansion of $(x - 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12}C_3 \left(x\right)^{12-3} \left(-2y\right)^3 = \left(-1\right)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot \left(2\right)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot \left(2\right)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$.

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$$\begin{aligned}
T_{13} = T_{12+1} &= {}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12} \\
&= (-1)^{12}\frac{18!}{12!6!}(9)^6(x)^6\left(\frac{1}{3}\right)^{12}\left(\frac{1}{\sqrt{x}}\right)^{12} \\
&= \frac{18\cdot17\cdot16\cdot15\cdot14\cdot13.12!}{12!.6\cdot5\cdot4\cdot3\cdot2}\cdot x^6.\left(\frac{1}{x^6}\right).3^{12}\left(\frac{1}{3^{12}}\right) \qquad \left[9^6 = \left(3^2\right)^6 = 3^{12}\right] \\
&= 18564
\end{aligned}$$

Question 7:

Find the middle terms in the expansions of $\left(3 - \frac{x^3}{6}\right)^7$

It is known that in the expansion of $(a + b)^n$, if n is odd, then there are two middle terms, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th} = 4^{th}$ term and $\left(\frac{7+1}{2}+1\right)^{th} = 5^{th}$ term

$$\begin{aligned}
T_4 = T_{3+1} &= {}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\frac{7!}{3!4!}.3^4\cdot\frac{x^9}{6^3} \\
&= -\frac{7\cdot6\cdot5.4!}{3\cdot2.4!}\cdot3^4\cdot\frac{1}{2^3\cdot3^3}\cdot x^9 = -\frac{105}{8}x^9 \\
T_5 = T_{4+1} &= {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4 = (-1)^4\frac{7!}{4!3!}(3)^3\cdot\frac{x^{12}}{6^4} \\
&= \frac{7\cdot6\cdot5.4!}{4!.3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^{12} = \frac{35}{48}x^{12}
\end{aligned}$$

Thus, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$.

Question 8:

Find the middle terms in the expansions of $\left(\frac{x}{3}+9y\right)^{10}$

It is known that in the expansion $(a + b)^n$, if n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th} = 6^{th}$ term

$$T_6 = T_{5+1} = {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = \frac{10!}{5!5!}\cdot\frac{x^5}{3^5}\cdot 9^5\cdot y^5$$

$$= \frac{10.9\cdot 8\cdot 7\cdot 6.5!}{5\cdot 4\cdot 3\cdot 2.5!}\cdot\frac{1}{3^5}\cdot 3^{10}\cdot x^5y^5 \qquad \left[9^5 = \left(3^2\right)^5 = 3^{10}\right]$$

$$= 252\times 3^5\cdot x^5\cdot y^5 = 61236x^5y^5$$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is 61236 x^5y^5.

Question 9:

In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that a^m occurs in the $(r + 1)^{th}$ term of the expansion $(1 + a)^{m+n}$, we obtain

$$T_{r+1} = {}^{m+n}C_r(1)^{m+n-r}(a)^r = {}^{m+n}C_r a^r$$

Comparing the indices of a in a^m and in T_{r+1}, we obtain

$r = m$

Therefore, the coefficient of a^m is

$$^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \qquad ...(1)$$

Assuming that a^n occurs in the $(k+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$$

Comparing the indices of a in a^n and in T_{k+1}, we obtain

$k = n$

Therefore, the coefficient of a^n is

$$^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \qquad ...(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of

$(x+1)^n$ are in the ratio 1:3:5. Find n and r.

It is known that $(k+1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a+b)^n$ is given by $T_{k+1} = {}^nC_k a^{n-k} b^k$.

Therefore, $(r-1)^{th}$ term in the expansion of $(x+1)^n$ is $T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2} x^{n-r+2}$

r^{th} term in the expansion of $(x+1)^n$ is $T_r = {}^nC_{r-1} (x)^{n-(r-1)} (1)^{(r-1)} = {}^nC_{r-1} x^{n-r+1}$

$(r+1)^{th}$ term in the expansion of $(x+1)^n$ is $T_{r+1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_r x^{n-r}$

Therefore, the coefficients of the $(r-1)^{th}$, r^{th}, and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are ${}^nC_{r-2}$, ${}^nC_{r-1}$, and ${}^nC_r$ respectively. Since these coefficients are in the ratio 1:3:5, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!}$$

$$= \frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r-3 = n-r+2$$

$$\Rightarrow n-4r+5=0 \qquad ...(1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{r}{n-r+1}$$

$$\therefore \frac{r}{n-r+1} = \frac{3}{5}$$

$$\Rightarrow 5r = 3n-3r+3$$

$$\Rightarrow 3n-8r+3=0 \qquad ...(2)$$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r - 12 = 0$

$\Rightarrow r = 3$

Putting the value of r in (1), we obtain

$n - 12 + 5 = 0$

$\Rightarrow n = 7$

Thus, $n = 7$ and $r = 3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^n occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$$T_{r+1} = {}^{2n}C_r (1)^{2n-r} (x)^r = {}^{2n}C_r (x)^r$$

Comparing the indices of x in x^n and in T_{r+1}, we obtain

$r = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is

$${}^{2n}C_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \qquad ...(1)$$

Assuming that x^n occurs in the $(k + 1)^{th}$ term of the expansion $(1 + x)^{2n-1}$, we obtain

$$T_{k+1} = {}^{2n-1}C_k (1)^{2n-1-k} (x)^k = {}^{2n-1}C_k (x)^k$$

Comparing the indices of x in x^n and T_{k+1}, we obtain

$k = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$ is

$${}^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!}$$

$$= \frac{2n.(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2.n!n!} = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \qquad ...(2)$$

From (1) and (2), it is observed that

$$\frac{1}{2}\left({}^{2n}C_n\right) = {}^{2n-1}C_n$$
$$\Rightarrow {}^{2n}C_n = 2\left({}^{2n-1}C_n\right)$$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Hence, proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion

$(1 + x)^m$ is 6.

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^2 occurs in the $(r + 1)^{th}$ term of the expansion $(1 + x)^m$, we obtain

$$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$.

It is given that the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

$\therefore {}^m C_2 = 6$

$\Rightarrow \frac{m!}{2!(m-2)!} = 6$

$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$

$\Rightarrow m(m-1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m(m-4) + 3(m-4) = 0$

$\Rightarrow (m-4)(m+3) = 0$

$\Rightarrow (m-4) = 0 \text{ or } (m+3) = 0$

$\Rightarrow m = 4 \text{ or } m = -3$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion

$(1 + x)^m$ is 6, is 4.