EXERCISE:-13.1

Question 1:

Evaluate the Given limit: $\lim_{x\to 3} x+3$

$$\lim_{x\to 3} x+3 = 3+3 = 6$$

Question 2:

Evaluate the Given limit: $\lim_{x\to\pi}\left(x-\frac{22}{7}\right)$

$$\lim_{x\to\pi}\left(x-\frac{22}{7}\right)=\left(\pi-\frac{22}{7}\right)$$

Question 3:

Evaluate the Given limit: $\lim_{r\to 1}\pi r^2$

$$\lim_{r\to 1}\pi r^2 = \pi(1)^2 = \pi$$

Question 4:

Evaluate the Given limit: $\lim_{x\to 4}\frac{4x+3}{x-2}$

$$\lim_{x\to 4}\frac{4x+3}{x-2}=\frac{4(4)+3}{4-2}=\frac{16+3}{2}=\frac{19}{2}$$

Question 5:

Evaluate the Given limit: $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

$$\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}=\frac{(-1)^{10}+(-1)^5+1}{-1-1}=\frac{1-1+1}{-2}=-\frac{1}{2}$$

Question 6:

Evaluate the Given limit: $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

$$\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$$

Put $x + 1 = y$ so that $y \to 1$ as $x \to 0$.

$$\begin{aligned}
\text{Accordingly, } \lim_{x \to 0} \frac{(x+1)^5 - 1}{x} &= \lim_{y \to 1} \frac{y^5 - 1}{y - 1} \\
&= \lim_{y \to 1} \frac{y^5 - 1^5}{y - 1} \\
&= 5.1^{5-1} \qquad \left[\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right] \\
&= 5
\end{aligned}$$

$$\therefore \lim_{x \to 0} \frac{(x+5)^5 - 1}{x} = 5$$

Question 7:

Evaluate the Given limit: $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$

At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\begin{aligned}
\therefore \lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4} &= \lim_{x \to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)} \\
&= \lim_{x \to 2} \frac{3x+5}{x+2} \\
&= \frac{3(2)+5}{2+2} \\
&= \frac{11}{4}
\end{aligned}$$

Question 8:

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$
$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$
$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$
$$=\frac{6\times 18}{7}$$
$$=\frac{108}{7}$$

Question 9:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}=b$$

Question 10:

Evaluate the Given limit: $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At $z = 1$, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}} = x$ so that $z \to 1$ as $x \to 1$.

$$\text{Accordingly, } \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = \lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1}\frac{x^2-1^2}{x-1}$$

$$= 2.1^{2-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 2$$

$$\therefore \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = 2$$

Question 11:

Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c \neq 0$

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a} = \frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$

$$= \frac{a+b+c}{a+b+c}$$

$$= 1 \qquad [a+b+c \neq 0]$$

Question 12:

Evaluate the Given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At $x = -2$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$$
$$=\lim_{x\to -2}\frac{1}{2x}$$
$$=\frac{1}{2(-2)}=\frac{-1}{4}$$

Question 13:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

$$\lim_{x\to 0}\frac{\sin ax}{bx}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$$
$$=\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\left(\frac{a}{b}\right)$$
$$=\frac{a}{b}\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right) \qquad [x\to 0 \Rightarrow ax\to 0]$$
$$=\frac{a}{b}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$
$$=\frac{a}{b}$$

Question 14:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

$$=\left(\frac{a}{b}\right)\times\frac{\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{bx\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l} x\to 0\Rightarrow ax\to 0 \\ \text{and } x\to 0\Rightarrow bx\to 0\end{array}\right]$$

$$=\left(\frac{a}{b}\right)\times\frac{1}{1} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{a}{b}$$

Question 15:

Evaluate the Given limit: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that $x\to\pi\Rightarrow(\pi-x)\to 0$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

$$=\frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{1}{\pi}$$

Question 16:

Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi-x}$

$$\lim_{x\to 0}\frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0}=\frac{1}{\pi}$$

Question 17:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1} \qquad \left[\cos x = 1-2\sin^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Question 18:

Evaluate the Given limit: $\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x}$

$$\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\begin{aligned}
\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x} &= \frac{1}{b}\lim_{x \to 0} \frac{x(a + \cos x)}{\sin x} \\
&= \frac{1}{b}\lim_{x \to 0}\left(\frac{x}{\sin x}\right) \times \lim_{x \to 0}(a + \cos x) \\
&= \frac{1}{b} \times \frac{1}{\left(\lim_{x \to 0} \frac{\sin x}{x}\right)} \times \lim_{x \to 0}(a + \cos x) \\
&= \frac{1}{b} \times (a + \cos 0) \qquad \left[\lim_{x \to 0} \frac{\sin x}{x} = 1\right] \\
&= \frac{a+1}{b}
\end{aligned}$$

Question 19:

Evaluate the Given limit: $\lim_{x \to 0} x\sec x$

$$\lim_{x \to 0} x\sec x = \lim_{x \to 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

Question 20:

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$ $a, b, a + b \neq 0$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx}$$
$$= \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax + bx}{ax + bx\left(\frac{\sin bx}{bx}\right)}$$
$$= \frac{\left(\lim_{ax\to 0}\frac{\sin ax}{ax}\right)\times\lim_{x\to 0}(ax) + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx\left(\lim_{bx\to 0}\frac{\sin bx}{bx}\right)} \qquad [\text{As } x\to 0 \Rightarrow ax\to 0 \text{ and } bx\to 0]$$
$$= \frac{\lim_{x\to 0}(ax) + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$
$$= \frac{\lim_{x\to 0}(ax + bx)}{\lim_{x\to 0}(ax + bx)}$$
$$= \lim_{x\to 0}(1)$$
$$= 1$$

Question 21:

Evaluate the Given limit: $\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$

At $x = 0$, the value of the given function takes the form $\infty - \infty$.

Now,

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$$
$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$
$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$
$$= \lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$
$$= \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}$$
$$= \frac{0}{1} \qquad \left[\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$
$$= 0$$

Question 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x = \frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x - \frac{\pi}{2} = y$ so that $x \to \frac{\pi}{2},\ y \to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(\pi+2y)}{y}$$

$$= \lim_{y\to 0} \frac{\tan 2y}{y} \qquad [\tan(\pi+2y) = \tan 2y]$$

$$= \lim_{y\to 0} \frac{\sin 2y}{y\cos 2y}$$

$$= \lim_{y\to 0}\left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}\right)$$

$$= \left(\lim_{2y\to 0} \frac{\sin 2y}{2y}\right) \times \lim_{y\to 0}\left(\frac{2}{\cos 2y}\right) \qquad [y\to 0 \Rightarrow 2y\to 0]$$

$$= 1\times\frac{2}{\cos 0} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 1\times\frac{2}{1}$$

$$= 2$$

Question 23:

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$

The given function is

$$f(x) = \begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}[2x+3] = 2(0)+3 = 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$
$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$$

Question 24:

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$

The given function is

$$f(x) = \begin{cases} x^2-1, \; x \le 1 \\ -x^2-1, \; x > 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left[x^2-1\right] = 1^2-1 = 1-1 = 0$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}\left[-x^2-1\right] = -1^2-1 = -1-1 = -2$$

It is observed that $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$.

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question 25:

Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

The given function is

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$$

$$= \lim_{x\to 0}\left(\frac{-x}{x}\right) \qquad [\text{When } x \text{ is negaitve, } |x| = -x]$$

$$= \lim_{x\to 0}(-1)$$

$$= -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad [\text{When } x \text{ is positive, } |x| = x]$$

$$= \lim_{x\to 0}(1)$$

$$= 1$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 26:

Find $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

The given function is

$$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left[\frac{x}{|x|}\right]\\ &= \lim_{x\to 0}\left[\frac{x}{-x}\right] \qquad \left[\text{When } x<0,\ |x|=-x\right]\\ &= \lim_{x\to 0}(-1)\\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]\\ &= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad \left[\text{When } x>0,\ |x|=x\right]\\ &= \lim_{x\to 0}(1)\\ &= 1\end{aligned}$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 27:

Find $\lim_{x\to 5} f(x)$, where $f(x) = |x| - 5$

The given function is $f(x) = |x| - 5$.

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}\left[|x|-5\right]$

$= \lim_{x\to 5}(x-5)$ $\quad\left[\text{When } x>0,\ |x|=x\right]$

$= 5-5$

$= 0$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\left(|x|-5\right)$

$= \lim_{x\to 5}(x-5)$ $\quad\left[\text{When } x>0,\ |x|=x\right]$

$= 5-5$

$= 0$

$\therefore \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 0$

Hence, $\lim_{x\to 5} f(x) = 0$

Question 28:

Suppose $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$ and if $\lim_{x\to 1} f(x) = f(1)$ what are possible values of a and b?

The given function is

$$f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(a+bx) = a+b$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(b-ax) = b-a$

$f(1) = 4$

It is given that $\lim_{x\to 1} f(x) = f(1)$.

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$

$\Rightarrow a+b=4$ and $b-a=4$

On solving these two equations, we obtain $a=0$ and $b=4$.

Thus, the respective possible values of a and b are 0 and 4.

Question 29:

Let $a_1, a_2, \dots, a_n$ be fixed real numbers and define a function

$f(x)=(x-a_1)(x-a_2)...(x-a_n)$

What is $\lim\limits_{x\to a_1} f(x)$? For some $a \neq a_1, a_2..., a_n,$ compute $\lim\limits_{x\to a} f(x)$.

The given function is $f(x)=(x-a_1)(x-a_2)...(x-a_n)$

$$\begin{aligned}\lim_{x\to a_1} f(x) &= \lim_{x\to a_1}\left[(x-a_1)(x-a_2)...(x-a_n)\right]\\ &=\left[\lim_{x\to a_1}(x-a_1)\right]\left[\lim_{x\to a_1}(x-a_2)\right]...\left[\lim_{x\to a_1}(x-a_n)\right]\\ &=(a_1-a_1)(a_1-a_2)...(a_1-a_n)=0\end{aligned}$$

$$\therefore \lim_{x\to a_1} f(x)=0$$

$$\begin{aligned}\text{Now, } \lim_{x\to a} f(x) &= \lim_{x\to a}\left[(x-a_1)(x-a_2)...(x-a_n)\right]\\ &=\left[\lim_{x\to a}(x-a_1)\right]\left[\lim_{x\to a}(x-a_2)\right]...\left[\lim_{x\to a}(x-a_n)\right]\\ &=(a-a_1)(a-a_2).....(a-a_n)\end{aligned}$$

$$\therefore \lim_{x\to a} f(x)=(a-a_1)(a-a_2)...(a-a_n)$$

Question 30:

If $f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$.

For what value (s) of a does $\lim\limits_{x\to a} f(x)$ exists?

The given function is

$$f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$$

When $a=0$,

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}(|x|+1)\\ &= \lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x|=-x]\\ &= -0+1\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}(|x|-1)\\ &= \lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x|=x]\\ &= 0-1\\ &= -1\end{aligned}$$

Here, it is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When $a<0$,

$$\begin{aligned}\lim_{x\to a^-} f(x) &= \lim_{x\to a^-}(|x|+1)\\ &= \lim_{x\to a}(-x+1) \qquad [x<a<0 \Rightarrow |x|=-x]\\ &= -a+1\end{aligned}$$

$$\begin{aligned}\lim_{x\to a^+} f(x) &= \lim_{x\to a^+}(|x|+1)\\ &= \lim_{x\to a}(-x+1) \qquad [a<x<0 \Rightarrow |x|=-x]\\ &= -a+1\end{aligned}$$

$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = -a+1$

Thus, limit of $f(x)$ exists at $x=a$, where $a<0$.

When $a>0$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x|-1)$$
$$= \lim_{x\to a}(x-1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a-1$$
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x|-1)$$
$$= \lim_{x\to a}(x-1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a-1$$
$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x\to a} f(x)$ exists for all $a \neq 0$.

Question 31:

If the function $f(x)$ satisfies $\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$$
$$\Rightarrow \frac{\lim_{x\to 1}(f(x)-2)}{\lim_{x\to 1}(x^2-1)} = \pi$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi \lim_{x\to 1}(x^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi(1^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - \lim_{x\to 1} 2 = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - 2 = 0$$
$$\therefore \lim_{x\to 1} f(x) = 2$$

Question 32:

If $f(x) = \begin{cases} mx^2+n, & x<0 \\ nx+m, & 0 \le x \le 1 \\ nx^3+m, & x>1 \end{cases}$. For what integers m and n does $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist?

The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\left(mx^2+n\right)$$
$$=m(0)^2+n$$
$$=n$$
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}\left(nx+m\right)$$
$$=n(0)+m$$
$$=m.$$

Thus, $\lim\limits_{x\to 0} f(x)$ exists if $m = n$.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}\left(nx+m\right)$$
$$=n(1)+m$$
$$=m+n$$
$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}\left(nx^3+m\right)$$
$$=n(1)^3+m$$
$$=m+n$$
$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x).$$

Thus, $\lim\limits_{x\to 1} f(x)$ exists for any integral value of m and n.

EXERCISE:-13.2

Question 1:

Find the derivative of $x^2 - 2$ at $x = 10$.

Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$
$$= \lim_{h\to 0}\frac{\left[(10+h)^2-2\right]-\left(10^2-2\right)}{h}$$
$$= \lim_{h\to 0}\frac{10^2+2.10.h+h^2-2-10^2+2}{h}$$
$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$
$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2:

Find the derivative of $99x$ at $x = 100$.

Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$
$$= \lim_{h\to 0}\frac{99(100+h)-99(100)}{h}$$
$$= \lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}$$
$$= \lim_{h\to 0}\frac{99h}{h}$$
$$= \lim_{h\to 0}(99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question 3:

Find the derivative of x at $x = 1$.

Let $f(x) = x$. Accordingly,

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&= \lim_{h\to 0}\frac{(1+h)-1}{h}\\
&= \lim_{h\to 0}\frac{h}{h}\\
&= \lim_{h\to 0}(1)\\
&= 1
\end{aligned}
$$

Thus, the derivative of x at $x = 1$ is 1.

Question 4:

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(ii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h}\\
&= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}\\
&= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}\\
&= \lim_{h\to 0}\left(h^2+3x^2+3xh\right)\\
&= 0+3x^2+0=3x^2
\end{aligned}
$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2\right) - \left(x^2 - 2x - x + 2\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(hx + hx + h^2 - 2h - h\right)}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (2x + h - 3)$$

$$= (2x + 0 - 3)$$

$$= 2x - 3$$

(iii) Let $f(x) = \frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2 (x+h)^2} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2 (x+h)^2} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2 (x+h)^2} \right]$$

$$= \lim_{h \to 0} \left[\frac{-h - 2x}{x^2 (x+h)^2} \right]$$

$$= \frac{0 - 2x}{x^2 (x+0)^2} = \frac{-2}{x^3}$$

(iv) Let $f(x) = \frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1) - (x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

Question 5:

For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \ldots + \frac{x^2}{2} + x + 1$$

Prove that $f'(1) = 100 f'(0)$

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+...+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+...+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+...+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+...+x+1$$

At $x=0$,

$$f'(0)=1$$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+...+1+1=[1+1+...+1+1]_{100\text{ terms}}=1\times 100=100$$

Thus, $f'(1)=100\times f^1(0)$

Question 6:

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$ for some fixed real number a.

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$

$$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n\right)$$

$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+...+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}+a^n(0)$$

$$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$$

Question 7:

For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\frac{x-a}{x-b}$

(i) Let $f(x) = (x-a)(x-b)$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}\left(x^2 - (a+b)x + ab\right)$$

$$= \frac{d}{dx}\left(x^2\right) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let $f(x) = \left(ax^2 + b\right)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}\left(a^2x^4 + 2abx^2 + b^2\right) = a^2\frac{d}{dx}\left(x^4\right) + 2ab\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(b^2\right)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$f'(x) = a^2\left(4x^3\right) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax\left(ax^2 + b\right)$$

(iii) Let $f(x) = \frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$
$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$
$$=\frac{x-b-x+a}{(x-b)^2}$$
$$=\frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.

Let $f(x)=\frac{x^n-a^n}{x-a}$

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$
$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$
$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x-\frac{3}{4}$ (ii) $(5x^3+3x-1)(x-1)$

(iii) $x^{-3}(5+3x)$ (iv) $x^5(3-6x^{-9})$

(v) $x^{-4}(3-4x^{-5})$ (vi) $\frac{2}{x+1}-\frac{x^2}{3x-1}$

(i) Let $f(x)=2x-\frac{3}{4}$

$$\begin{aligned}f'(x)&=\frac{d}{dx}\left(2x-\frac{3}{4}\right)\\&=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)\\&=2-0\\&=2\end{aligned}$$

(ii) Let $f(x)=(5x^3+3x-1)(x-1)$

By Leibnitz product rule,

$$\begin{aligned}f'(x)&=(5x^3+3x-1)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}(5x^3+3x-1)\\&=(5x^3+3x-1)(1)+(x-1)(5.3x^2+3-0)\\&=(5x^3+3x-1)+(x-1)(15x^2+3)\\&=5x^3+3x-1+15x^3+3x-15x^2-3\\&=20x^3-15x^2+6x-4\end{aligned}$$

(iii) Let $f(x)=x^{-3}(5+3x)$

By Leibnitz product rule,

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x) + (5+3x)\frac{d}{dx}\left(x^{-3}\right)$$
$$= x^{-3}(0+3) + (5+3x)\left(-3x^{-3-1}\right)$$
$$= x^{-3}(3) + (5+3x)\left(-3x^{-4}\right)$$
$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$
$$= -6x^{-3} - 15x^{-4}$$
$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$
$$= \frac{-3x^{-3}}{x}(2x+5)$$
$$= \frac{-3}{x^4}(5+2x)$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5\frac{d}{dx}\left(3-6x^{-9}\right) + \left(3-6x^{-9}\right)\frac{d}{dx}\left(x^5\right)$$
$$= x^5\left\{0-6(-9)x^{-9-1}\right\} + \left(3-6x^{-9}\right)\left(5x^4\right)$$
$$= x^5\left(54x^{-10}\right) + 15x^4 - 30x^{-5}$$
$$= 54x^{-5} + 15x^4 - 30x^{-5}$$
$$= 24x^{-5} + 15x^4$$
$$= 15x^4 + \frac{24}{x^5}$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}\left(3-4x^{-5}\right)+\left(3-4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$$
$$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\}+\left(3-4x^{-5}\right)(-4)x^{-4-1}$$
$$= x^{-4}\left(20x^{-6}\right)+\left(3-4x^{-5}\right)\left(-4x^{-5}\right)$$
$$= 20x^{-10}-12x^{-5}+16x^{-10}$$
$$= 36x^{-10}-12x^{-5}$$
$$= -\frac{12}{x^5}+\frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1}-\frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right)-\frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right]-\left[\frac{(3x-1)\frac{d}{dx}\left(x^2\right)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$
$$= \left[\frac{(x+1)(0)-2(1)}{(x+1)^2}\right]-\left[\frac{(3x-1)(2x)-\left(x^2\right)(3)}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2}-\left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2}-\left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of cos x from first principle.

Let $f(x)$ = cos x. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$= -\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$= -\cos x(0)-\sin x(1) \qquad \left[\lim_{h\to 0}\frac{1-\cos h}{h}=0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$= -\sin x$$

$$\therefore f'(x) = -\sin x$$

Question 11:

Find the derivative of the following functions:

(i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5 \sec x + 4 \cos x$

(iv) $\operatorname{cosec} x$ (v) $3\cot x + 5\operatorname{cosec} x$

(vi) $5\sin x - 6\cos x + 7$ (vii) $2\tan x - 7\sec x$

(i) Let $f(x) = \sin x \cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$
$$= \lim_{h \to 0} \frac{1}{2h}\left[2\sin(x+h)\cos(x+h) - 2\sin x \cos x\right]$$
$$= \lim_{h \to 0} \frac{1}{2h}\left[\sin 2(x+h) - \sin 2x\right]$$
$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right]$$
$$= \lim_{h \to 0} \cos(2x+h) . \lim_{h \to 0} \frac{\sin h}{h}$$
$$= \cos(2x+0).1$$
$$= \cos 2x$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.1.\frac{\sin x}{\cos x}$$
$$=\sec x\tan x$$

(iii) Let $f(x) = 5\sec x + 4\cos x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-\left[5\sec x+4\cos x\right]}{h}$$

$$=5\lim_{h\to 0}\frac{\left[\sec(x+h)-\sec x\right]}{h}+4\lim_{h\to 0}\frac{\left[\cos(x+h)-\cos x\right]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}\left[\cos x\cos h-\sin x\sin h-\cos x\right]$$

$$=\frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}\left[-\cos x(1-\cos h)-\sin x\sin h\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4\left[(-\cos x).(0)-(\sin x).1\right]$$

$$=\frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$

$$=\frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$=5\sec x\tan x.-4\sin x$$

(iv) Let $f(x)=\operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{3\cot(x+h)+5\operatorname{cosec}(x+h)-3\cot x-5\operatorname{cosec} x}{h}$$

$$=3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right] \qquad \ldots(1)$$

Now, $\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)$$

$$=-1.\frac{1}{\sin x\cdot\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \qquad \ldots(2)$$

$$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec}x\cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\{\sin(x+h)-\sin x\}-6\{\cos(x+h)-\cos x\}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$

$$=5\cos x+6\sin x$$

(vii) Let $f(x) = 2\tan x - 7\sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$