## Exercise-5.1

**Question 1:**

Prove that the function $f(x)=5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

The given function is $f(x)=5x-3$

At $x=0$, $f(0)=5\times0-3=3$

$\lim_{x\to0} f(x)=\lim_{x\to0}(5x-3)=5\times0-3=-3$

$\therefore \lim_{x\to0} f(x)=f(0)$

Therefore, $f$ is continuous at $x=0$

At $x=-3$, $f(-3)=5\times(-3)-3=-18$

$\lim_{x\to-3} f(x)=\lim_{x\to-3}(5x-3)=5\times(-3)-3=-18$

$\therefore \lim_{x\to-3} f(x)=f(-3)$

Therefore, $f$ is continuous at $x=-3$

At $x=5$, $f(x)=f(5)=5\times5-3=25-3=22$

$\lim_{x\to5} f(x)=\lim_{x\to5}(5x-3)=5\times5-3=22$

$\therefore \lim_{x\to5} f(x)=f(5)$

Therefore, $f$ is continuous at $x=5$

**Question 2:**

Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

The given function is $f(x)=2x^2-1$

At $x=3$, $f(x)=f(3)=2\times3^2-1=17$

$\lim_{x\to3} f(x)=\lim_{x\to3}(2x^2-1)=2\times3^2-1=17$

$\therefore \lim_{x\to3} f(x)=f(3)$

Thus, $f$ is continuous at $x = 3$

**Question 3:**

Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$ (d) $f(x) = |x - 5|$

(a) The given function is $f(x) = x - 5$

It is evident that $f$ is defined at every real number $k$ and its value at $k$ is $k - 5$.

It is also observed that, $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - 5 = f(k)$

$\therefore \lim_{x \to k} f(x) = f(k)$

Hence, $f$ is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we obtain

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x-5} = \frac{1}{k-5}$$

$$\text{Also, } f(k) = \frac{1}{k-5} \qquad (\text{As } k \neq 5)$$

$$\therefore \lim_{x \to k} f(x) = f(k)$$

Hence, $f$ is continuous at every point in the domain of $f$ and therefore, it is a continuous function.

(c) The given function is $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

For any real number $c \neq -5$, we obtain

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 25}{x + 5} = \lim_{x \to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Hence, $f$ is continuous at every point in the domain of $f$ and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, & \text{if } x < 5 \\ x-5, & \text{if } x \geq 5 \end{cases}$

This function $f$ is defined at all points of the real line.

Let $c$ be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(5-x) = 5-c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all real numbers less than 5.

Case II : $c = 5$

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5}(5-x) = (5-5) = 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5}(x-5) = 0$$

$$\therefore \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Therefore, $f$ is continuous at $x = 5$

Case III: $c > 5$

$$\text{Then, } f(c) = f(5) = c - 5$$
$$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers greater than 5.

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

**Question 4:**

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where $n$ is a positive integer.

The given function is $f(x) = x^n$

It is evident that *f* is defined at all positive integers, $n$, and its value at $n$ is $n^n$.

$$\text{Then, } \lim_{x \to n} f(n) = \lim_{x \to n} (x^n) = n^n$$

$$\therefore \lim_{x \to n} f(x) = f(n)$$

Therefore, *f* is continuous at $n$, where $n$ is a positive integer.

**Question 5:**

Is the function *f* defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

The given function *f* is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that $f$ is defined at 0 and its value at 0 is 0.

$$\text{Then, } \lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$$

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

Therefore, $f$ is continuous at $x = 0$

At $x = 1$,

$f$ is defined at 1 and its value at 1 is 1.

The left hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (5) = 5$$
$$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore, $f$ is not continuous at $x = 1$

At $x = 2$,

$f$ is defined at 2 and its value at 2 is 5.

$$\text{Then, } \lim_{x\to 2} f(x) = \lim_{x\to 2} (5) = 5$$

$$\therefore \lim_{x\to 2} f(x) = f(2)$$

Therefore, $f$ is continuous at $x = 2$

Question 6:

Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

The given function $f$ is $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$

It is evident that the given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line. Then, three cases arise.

(i) $c<2$

(ii) $c>2$

(iii) $c=2$

Case (i) $c<2$

$$\text{Then, } f(c)=2c+3$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x+3)=2c+3$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x<2$

Case (ii) $c>2$

$$\text{Then, } f(c)=2c-3$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x-3)=2c-3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x>2$

Case (iii) $c=2$

Then, the left hand limit of $f$ at $x=2$ is,

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)=2\times 2+3=7$$

The right hand limit of $f$ at $x=2$ is,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(2x-3) = 2\times 2-3 = 1$$

It is observed that the left and right hand limit of *f* at *x* = 2 do not coincide.

Therefore, *f* is not continuous at *x* = 2

Hence, *x* = 2 is the only point of discontinuity of *f*.

Question 7:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} |x|+3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x+2, \text{if } x \ge 3 \end{cases}$$

The given function *f* is $f(x) = \begin{cases} |x|+3 = -x+3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x+2, \text{ if } x \ge 3 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c < -3, \text{ then } f(c) = -c+3$$
$$\lim_{x\to c} f(x) = \lim_{x\to c}(-x+3) = -c+3$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < −3

Case II:

$$\text{If } c = -3, \text{ then } f(-3) = -(-3)+3 = 6$$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-}(-x+3) = -(-3)+3 = 6$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}(-2x) = -2\times(-3) = 6$$
$$\therefore \lim_{x\to -3} f(x) = f(-3)$$

Therefore, $f$ is continuous at $x = -3$

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(-2x) = -2c$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, $f$ is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of $f$ at $x = 3$ is,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(-2x) = -2\times 3 = -6$$

The right hand limit of $f$ at $x = 3$ is,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(6x+2) = 6\times 3+2 = 20$$

It is observed that the left and right hand limit of $f$ at $x = 3$ do not coincide.

Therefore, $f$ is not continuous at $x = 3$

Case V:

If $c > 3$, then $f(c) = 6c+2$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(6x+2) = 6c+2$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of $f$.

**Question 8:**

Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x)=\begin{cases}\frac{|x|}{x} \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$$

The given function $f$ is $$f(x)=\begin{cases}\frac{|x|}{x} \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$$

It is known that, $x<0 \Rightarrow |x|=-x$ and $x>0 \Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 \text{ if } x<0\\ 0, \text{ if } x=0\\ \frac{|x|}{x}=\frac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:

$$\text{If } c<0, \text{ then } f(c)=-1$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, $f$ is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of $f$ at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of $f$ at $x = 0$ is,

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(1) = 1$$

It is observed that the left and right hand limit of $f$ at $x = 0$ do not coincide.

Therefore, $f$ is not continuous at $x = 0$

Case III:

$$\text{If } c > 0,\ \text{then } f(c) = 1$$
$$\lim_{x\to c} f(x) = \lim_{x\to c}(1) = 1$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of $f$.

**Question 9:**

Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

The given function $f$ is $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{x}{|x|}=\frac{x}{-x}=-1, \text{ if } x<0\\ -1, \text{ if } x\geq 0\end{cases}$$

$$\Rightarrow f(x)=-1 \text{ for all } x\in \mathbf{R}$$

Let $c$ be any real number. Then, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-1)=-1$

Also, $f(c)=-1=\lim\limits_{x\to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x)=\begin{cases}x+1, \text{ if } x\geq 1\\ x^2+1, \text{ if } x<1\end{cases}$$

The given function $f$ is $f(x)=\begin{cases}x+1, \text{ if } x\geq 1\\ x^2+1, \text{ if } x<1\end{cases}$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^2+1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x^2+1)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x < 1$

Case II:

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(x^2+1\right) = 1^2+1 = 2$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 1+1 = 2$$
$$\therefore \lim_{x\to 1} f(x) = f(1)$$

Therefore, $f$ is continuous at $x = 1$

Case III:

$$\text{If } c > 1, \text{ then } f(c) = c+1$$
$$\lim_{x\to c} f(x) = \lim_{x\to c}(x+1) = c+1$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x > 1$

Hence, the given function $f$ has no point of discontinuity.

**Question 11:**

Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} x^3-3, & \text{if } x \le 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$$

The given function $f$ is $f(x) = \begin{cases} x^3-3, & \text{if } x \le 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x^3 - 3) = c^3 - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x < 2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(x^3 - 3) = 2^3 - 3 = 5$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(x^2 + 1) = 2^2 + 1 = 5$$
$$\therefore \lim_{x \to 2} f(x) = f(2)$$

Therefore, *f* is continuous at $x = 2$

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(x^2 + 1) = c^2 + 1$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x > 2$

Thus, the given function *f* is continuous at every point on the real line.

Hence, *f* has no point of discontinuity.

**Question 12:**

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

The given function $f$ is $f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^{10}-1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x < 1$

Case II:

If $c = 1$, then the left hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^{2}\right)=1^2=1$$

It is observed that the left and right hand limit of $f$ at $x = 1$ do not coincide.

Therefore, $f$ is not continuous at $x = 1$

Case III:

If $c>1$, then $f(c)=c^2$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{2}\right)=c^2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of $f$.

**Question 13:**

Is the function defined by

$$f(x)=\begin{cases}x+5, & \text{if } x\le 1\\ x-5, & \text{if } x>1\end{cases}$$

a continuous function?

The given function is $f(x)=\begin{cases}x+5, & \text{if } x\le 1\\ x-5, & \text{if } x>1\end{cases}$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c+5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+5)=c+5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x < 1$

Case II:

$$\text{If } c=1, \text{ then } f(1)=1+5=6$$

The left hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)=1+5=6$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)=1-5=-4$$

It is observed that the left and right hand limit of $f$ at $x = 1$ do not coincide.

Therefore, $f$ is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c - 5$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c - 5$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of $f$.

**Question 14:**

Discuss the continuity of the function $f$, where $f$ is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

The given function is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let $c$ be a point in the interval [0, 10].

Case I:

If $0 \le c < 1$, then $f(c) = 3$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(3) = 3$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, $f$ is continuous in the interval [0, 1).

Case II:

If $c = 1$, then $f(3) = 3$

The left hand limit of $f$ at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$$

It is observed that the left and right hand limits of $f$ at $x = 1$ do not coincide.

Therefore, $f$ is not continuous at $x = 1$

Case III:

If $1 < c < 3$, then $f(c) = 4$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, $f$ is continuous at all points of the interval (1, 3).

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of $f$ at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$$

The right hand limit of $f$ at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$$

It is observed that the left and right hand limits of $f$ at $x = 3$ do not coincide.

Therefore, $f$ is not continuous at $x = 3$

Case V:

If $3 < c \leq 10$, then $f(c) = 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points of the interval (3, 10].

Hence, $f$ is not continuous at $x = 1$ and $x = 3$

**Question 15:**

Discuss the continuity of the function $f$, where $f$ is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let $c$ be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = 2c$$
$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x < 0$

Case II:

$$\text{If } c = 0, \text{ then } f(c) = f(0) = 0$$

The left hand limit of $f$ at $x = 0$ is,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x) = 2\times 0 = 0$

The right hand limit of *f* at $x = 0$ is,

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(0) = 0$
$\therefore \lim_{x\to 0} f(x) = f(0)$

Therefore, *f* is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(0) = 0$
$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, *f* is continuous at all points of the interval (0, 1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of *f* at $x = 1$ is,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(0) = 0$

The right hand limit of *f* at $x = 1$ is,

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x) = 4\times 1 = 4$

It is observed that the left and right hand limits of *f* at $x = 1$ do not coincide.

Therefore, *f* is not continuous at $x = 1$

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(4x) = 4c$
$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x > 1$

Hence, $f$ is not continuous only at $x = 1$

**Question 16:**

Discuss the continuity of the function $f$, where $f$ is defined by

$$f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$$

The given function $f$ is $f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$

The given function is defined at all points of the real line.

Let $c$ be a point on the real line.

Case I:

If $c<-1$, then $f(c)=-2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x < -1$

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of $f$ at $x = -1$ is,

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}(-2)=-2$

The right hand limit of $f$ at $x = -1$ is,

$\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}(2x)=2\times(-1)=-2$

$\therefore \lim\limits_{x\to -1} f(x)=f(-1)$

Therefore, $f$ is continuous at $x = -1$

Case III:

$$\text{If } -1 < c < 1, \text{ then } f(c) = 2c$$
$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points of the interval $(-1, 1)$.

Case IV:

$$\text{If } c = 1, \text{ then } f(c) = f(1) = 2 \times 1 = 2$$

The left hand limit of $f$ at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(2x) = 2 \times 1 = 2$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

$$\therefore \lim_{x \to 1} f(x) = f(c)$$

Therefore, $f$ is continuous at $x = 2$

Case V:

$$\text{If } c > 1, \text{ then } f(c) = 2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(2) = 2$$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x > 1$

Thus, from the above observations, it can be concluded that $f$ is continuous at all points of the real line.

**Question 17:**

Find the relationship between $a$ and $b$ so that the function $f$ defined by

$$f(x)=\begin{cases} ax+1, & \text{if } x\le 3 \\ bx+3, & \text{if } x>3 \end{cases}$$

is continuous at $x = 3$.

The given function $f$ is $f(x)=\begin{cases} ax+1, & \text{if } x\le 3 \\ bx+3, & \text{if } x>3 \end{cases}$

If $f$ is continuous at $x = 3$, then

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3) \qquad ...(1)$$

Also,

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(ax+1)=3a+1$$
$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(bx+3)=3b+3$$
$$f(3)=3a+1$$

Therefore, from (1), we obtain

$$3a+1=3b+3=3a+1$$
$$\Rightarrow 3a+1=3b+3$$
$$\Rightarrow 3a=3b+2$$
$$\Rightarrow a=b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a=b+\frac{2}{3}$

**Question 18:**

For what value of $\lambda$ is the function defined by

$$f(x)=\begin{cases} \lambda\left(x^2-2x\right), & \text{if } x\le 0 \\ 4x+1, & \text{if } x>0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

The given function $f$ is $f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$

If $f$ is continuous at $x = 0$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$
$$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+}(4x+1) = \lambda(0^2-2\times 0)$$
$$\Rightarrow \lambda(0^2-2\times 0) = 4\times 0+1 = 0$$
$$\Rightarrow 0 = 1 = 0,\ \text{which is not possible}$$

Therefore, there is no value of λ for which $f$ is continuous at $x = 0$

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x\to 1}(4x+1) = 4\times 1+1 = 5$$
$$\therefore \lim_{x\to 1} f(x) = f(1)$$

Therefore, for any values of λ, $f$ is continuous at $x = 1$

**Question 19:**

Show that the function defined by $g(x)=x-[x]$ is discontinuous at all integral point. Here $[x]$ denotes the greatest integer less than or equal to $x$.

The given function is $g(x)=x-[x]$

It is evident that $g$ is defined at all integral points.

Let $n$ be an integer.

Then,

$$g(n)=n-[n]=n-n=0$$

The left hand limit of $f$ at $x = n$ is,

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x-[x]) = \lim_{x\to n^-}(x) - \lim_{x\to n^-}[x] = n-(n-1) = 1$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x-[x]) = \lim_{x\to n^+}(x) - \lim_{x\to n^+}[x] = n-n = 0$$

It is observed that the left and right hand limits of $f$ at $x = n$ do not coincide.

Therefore, $f$ is not continuous at $x = n$

Hence, $g$ is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x$ = p?

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that $f$ is defined at $x$ = p

$$\text{At } x = \pi,\ f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$$

$$\text{Consider } \lim_{x\to\pi} f(x) = \lim_{x\to\pi}(x^2 - \sin x + 5)$$

$$\text{Put } x = \pi + h$$

$$\text{If } x \to \pi, \textit{ then it is evident that } h \to 0$$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}(x^2 - \sin x + 5) \\
&= \lim_{h\to 0}\left[(\pi+h)^2 - \sin(\pi+h) + 5\right] \\
&= \lim_{h\to 0}(\pi+h)^2 - \lim_{h\to 0}\sin(\pi+h) + \lim_{h\to 0} 5 \\
&= (\pi+0)^2 - \lim_{h\to 0}\left[\sin\pi\cosh + \cos\pi\sinh\right] + 5 \\
&= \pi^2 - \lim_{h\to 0}\sin\pi\cosh - \lim_{h\to 0}\cos\pi\sinh + 5 \\
&= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5 \\
&= \pi^2 - 0\times 1 - (-1)\times 0 + 5 \\
&= \pi^2 + 5
\end{aligned}$$

$$\therefore \lim_{x\to\pi} f(x) = f(\pi)$$

Therefore, the given function $f$ is continuous at $x = \pi$

**Question 21:**

Discuss the continuity of the following functions.

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \times \cos x$

It is known that if $g$ and $h$ are two continuous functions, then

$g+h,\ g-h,\ \text{and } g.h$ are also continuous.

It has to proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\begin{aligned}
&g(c) = \sin c \\
&\lim_{x\to c} g(x) = \lim_{x\to c} \sin x \\
&\qquad = \lim_{h\to 0} \sin(c+h) \\
&\qquad = \lim_{h\to 0}\left[\sin c \cos h + \cos c \sin h\right] \\
&\qquad = \lim_{h\to 0}(\sin c \cos h) + \lim_{h\to 0}(\cos c \sin h) \\
&\qquad = \sin c \cos 0 + \cos c \sin 0 \\
&\qquad = \sin c + 0 \\
&\qquad = \sin c \\
&\therefore\ \lim_{x\to c} g(x) = g(c)
\end{aligned}$$

Therefore, $g$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\ &= \lim_{h\to 0} \cos(c+h) \\ &= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, $h$ is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

**Question 22:**

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

It is known that if $g$ and $h$ are two continuous functions, then

$(i)\ \frac{h(x)}{g(x)},\ g(x) \neq 0$ is continuous

$(ii)\ \frac{1}{g(x)},\ g(x) \neq 0$ is continuous

$(iii)\ \frac{1}{h(x)},\ h(x) \neq 0$ is continuous

It has to be proved first that *g* (*x*) = sin *x* and *h* (*x*) = cos *x* are continuous functions.

Let *g* (*x*) = sin *x*

It is evident that *g* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$
$$\lim_{x\to c} g(x) = \lim_{x\to c} \sin x$$
$$= \lim_{h\to 0} \sin(c+h)$$
$$= \lim_{h\to 0} [\sin c \cos h + \cos c \sin h]$$
$$= \lim_{h\to 0} (\sin c \cos h) + \lim_{h\to 0} (\cos c \sin h)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$
$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let *h* (*x*) = cos *x*

It is evident that *h* (*x*) = cos *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* ® *c*, then *h* ® 0

*h* (*c*) = cos *c*

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\ &= \lim_{h\to 0} \cos(c+h) \\ &= \lim_{h\to 0}[\cos c \cos h - \sin c \sin h] \\ &= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, *h* (*x*) = cos *x* is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ \ (n \in Z)\ \text{ is continuous}$$

Therefore, cosecant is continuous except at *x* = *n*p, *n* Î **Z**

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ \ (n \in \mathbf{Z})\ \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ \ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ \ (n \in Z)\ \text{ is continuous}$$

Therefore, cotangent is continuous except at *x* = *n*p, *n* Î **Z**

Question 23:

Find the points of discontinuity of *f*, where

$$f(x)=\begin{cases}\frac{\sin x}{x}, \text{ if } x<0\\ x+1, \text{ if } x\geq 0\end{cases}$$

The given function $f$ is $f(x)=\begin{cases}\frac{\sin x}{x}, \text{ if } x<0\\ x+1, \text{ if } x\geq 0\end{cases}$

It is evident that $f$ is defined at all points of the real line.

Let $c$ be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x < 0$

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x > 0$

Case III:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0+1=1$$

The left hand limit of $f$ at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of $f$ at $x = 0$ is,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 1$$
$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Therefore, $f$ is continuous at $x = 0$

From the above observations, it can be concluded that $f$ is continuous at all points of the real line.

Thus, $f$ has no point of discontinuity.

**Question 24:**

Determine if $f$ defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

The given function $f$ is $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

It is evident that $f$ is defined at all points of the real line.

Let $c$ be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c) = c^2 \sin\frac{1}{c}$$
$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c} \sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x \neq 0$

Case II:

If $c=0$, then $f(0)=0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(x^2\sin\frac{1}{x}\right)=\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)$$

It is known that, $-1\le\sin\frac{1}{x}\le 1,\ x\ne 0$

$$\Rightarrow -x^2\le\sin\frac{1}{x}\le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right)\le\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)\le\lim_{x\to 0}x^2$$

$$\Rightarrow 0\le\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)\le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)=0$$

$$\therefore \lim_{x\to 0^-} f(x)=0$$

Similarly, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left(x^2\sin\frac{1}{x}\right)=\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)=0$

$$\therefore \lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x)$$

Therefore, $f$ is continuous at $x = 0$

From the above observations, it can be concluded that $f$ is continuous at every point of the real line.

Thus, $f$ is a continuous function.

**Question 25:**

Examine the continuity of $f$, where $f$ is defined by

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\ne 0\\ -1 & \text{if } x=0\end{cases}$$

The given function $f$ is $f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\ne 0\\ -1 & \text{if } x=0\end{cases}$

It is evident that $f$ is defined at all points of the real line.

Let $c$ be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c) = \sin c - \cos c$$
$$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x \neq 0$

Case II:

$$\text{If } c = 0, \text{ then } f(0) = -1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Therefore, $f$ is continuous at $x = 0$

From the above observations, it can be concluded that $f$ is continuous at every point of the real line.

Thus, $f$ is a continuous function.

**Question 26:**

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

The given function $f$ is $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

The given function $f$ is continuous at $x=\frac{\pi}{2}$, if $f$ is defined at $x=\frac{\pi}{2}$ and if the value of the $f$ at $x=\frac{\pi}{2}$ equals the limit of $f$ at $x=\frac{\pi}{2}$.

It is evident that $f$ is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

$$\text{Put } x=\frac{\pi}{2}+h$$

$$\text{Then, } x\to\frac{\pi}{2}\Rightarrow h\to 0$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of $k$ is 6.

**Question 27:**

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases} \qquad \text{at } x=2$$

The given function is $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$

The given function $f$ is continuous at $x = 2$, if $f$ is defined at $x = 2$ and if the value of $f$ at $x = 2$ equals the limit of $f$ at $x = 2$

It is evident that $f$ is defined at $x = 2$ and $f(2)=k(2)^2=4k$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$
$$\Rightarrow \lim_{x\to 2^-} (kx^2) = \lim_{x\to 2^+} (3) = 4k$$
$$\Rightarrow k\times 2^2 = 3 = 4k$$
$$\Rightarrow 4k = 3 = 4k$$
$$\Rightarrow 4k = 3$$
$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of $k$ is $\frac{3}{4}$.

**Question 28:**

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function $f$ is continuous at $x$ = p, if $f$ is defined at $x$ = p and if the value of $f$ at $x$ = pequals the limit of $f$ at $x$ = p

It is evident that $f$ is defined at $x$ = p and $f(\pi) = k\pi + 1$

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x) = f(\pi)$$
$$\Rightarrow \lim_{x\to \pi^-} (kx+1) = \lim_{x\to \pi^+} \cos x = k\pi + 1$$
$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$
$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$
$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of $k$ is $-\frac{2}{\pi}$.

**Question 29:**

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases} \quad \text{at } x=5$$

The given function $f$ is $f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$

The given function $f$ is continuous at $x$ = 5, if $f$ is defined at $x$ = 5 and if the value of $f$ at $x$ = 5 equals the limit of $f$ at $x$ = 5

It is evident that $f$ is defined at $x$ = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of $k$ is $\frac{9}{5}$.

**Question 30:**

Find the values of $a$ and $b$ such that the function defined by

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

is a continuous function.

The given function $f$ is $f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$

It is evident that the given function $f$ is defined at all points of the real line.

If $f$ is a continuous function, then $f$ is continuous at all real numbers.

In particular, $f$ is continuous at $x = 2$ and $x = 10$

Since $f$ is continuous at $x = 2$, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since $f$ is continuous at $x = 10$, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

By putting $a = 2$ in equation (1), we obtain

$2 \times 2 + b = 5$

$\Rightarrow 4 + b = 5$

$\Rightarrow b = 1$

Therefore, the values of $a$ and $b$ for which $f$ is a continuous function are 2 and 1 respectively.

**Question 31:**

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

The given function is $f(x) = \cos(x^2)$

This function $f$ is defined for every real number and $f$ can be written as the composition of two functions as,

$f = g\ o\ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that $g$ is defined for every real number.

Let $c$ be a real number.

Then, $g(c) = \cos c$

$$\begin{aligned}
&\text{Put } x = c + h \\
&\text{If } x \to c, \text{ then } h \to 0 \\
&\lim_{x\to c} g(x) = \lim_{x\to c} \cos x \\
&\qquad = \lim_{h\to 0} \cos(c+h) \\
&\qquad = \lim_{h\to 0} \left[\cos c \cos h - \sin c \sin h\right] \\
&\qquad = \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\
&\qquad = \cos c \cos 0 - \sin c \sin 0 \\
&\qquad = \cos c \times 1 - \sin c \times 0 \\
&\qquad = \cos c \\
&\therefore \lim_{x\to c} g(x) = g(c)
\end{aligned}$$

Therefore, $g(x) = \cos x$ is continuous function.

$h(x) = x^2$

Clearly, $h$ is defined for every real number.

Let $k$ be a real number, then $h(k) = k^2$

$$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$$
$$\therefore \lim_{x \to k} h(x) = h(k)$$

Therefore, $h$ is a continuous function.

It is known that for real valued functions $g$ and $h$,such that ($g$ o $h$) is defined at $c$, if $g$ is continuous at $c$ and if $f$ is continuous at $g$ ($c$), then ($f$ o $g$) is continuous at $c$.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

**Question 32:**

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

The given function is $f(x) = |\cos x|$

This function $f$ is defined for every real number and $f$ can be written as the composition of two functions as,

$f = g$ o $h$, where $g(x) = |x|$ and $h(x) = \cos x$

$$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, $g$ is defined for all real numbers.

Let $c$ be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c}(-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, $g$ is continuous at all points $x$, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, $g$ is continuous at all points $x$, such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-}(-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+}(x) = 0$

$\therefore \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+}(x) = g(0)$

Therefore, $g$ is continuous at $x = 0$

From the above three observations, it can be concluded that $g$ is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x\to c} h(x) = \lim_{x\to c} \cos x$$
$$= \lim_{h\to 0} \cos(c+h)$$
$$= \lim_{h\to 0}\left[\cos c\cos h - \sin c\sin h\right]$$
$$= \lim_{h\to 0} \cos c\cos h - \lim_{h\to 0} \sin c\sin h$$
$$= \cos c\cos 0 - \sin c\sin 0$$
$$= \cos c\times 1 - \sin c\times 0$$
$$= \cos c$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions $g$ and $h$,such that ($g$ o $h$) is defined at $c$, if $g$ is continuous at $c$ and if $f$ is continuous at $g$ ($c$), then ($f$ o $g$) is continuous at $c$.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Let $f(x) = \sin|x|$

This function $f$ is defined for every real number and $f$ can be written as the composition of two functions as,

$f = g$ o $h$, where $g(x) = |x|$ and $h(x) = \sin x$

$$\left[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)\right]$$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, $g$ is defined for all real numbers.

Let $c$ be a real number.

Case I:

$$\text{If } c < 0, \text{ then } g(c) = -c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c}(-x) = -c$$
$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g$ is continuous at all points $x$, such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } g(c) = c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} x = c$$
$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g$ is continuous at all points $x$, such that $x > 0$

Case III:

$$\text{If } c = 0, \text{ then } g(c) = g(0) = 0$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+}(x) = g(0)$$

Therefore, $g$ is continuous at $x = 0$

From the above three observations, it can be concluded that $g$ is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h(c) = \sin c$

$h(c) = \sin c$

$$\begin{aligned}\lim_{x \to c} h(x) &= \lim_{x \to c} \sin x \\ &= \lim_{k \to 0} \sin(c+k) \\ &= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k] \\ &= \lim_{k \to 0} (\sin c \cos k) + \lim_{h \to 0} (\cos c \sin k) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$

$\therefore \lim_{x \to c} h(x) = g(c)$

Therefore, $h$ is a continuous function.

It is known that for real valued functions $g$ and $h$,such that ($g$ o $h$) is defined at $c$, if $g$ is continuous at $c$ and if $f$ is continuous at $g$ ($c$), then ($f$ o $g$) is continuous at $c$.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of $f$ defined by $f(x) = |x| - |x+1|$.

The given function is $f(x) = |x| - |x+1|$

The two functions, $g$ and $h$, are defined as

$g(x) = |x|$ and $h(x) = |x+1|$

Then, $f = g - h$

The continuity of $g$ and $h$ is examined first.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, $g$ is defined for all real numbers.

Let $c$ be a real number.

Case I:

$$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, $g$ is continuous at all points $x$, such that $x < 0$

Case II:

$$\text{If } c>0, \text{ then } g(c)=c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c} x=c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, $g$ is continuous at all points $x$, such that $x > 0$

Case III:

$$\text{If } c=0, \text{ then } g(c)=g(0)=0$$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$
$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$
$$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$$

Therefore, $g$ is continuous at $x = 0$

From the above three observations, it can be concluded that $g$ is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), & \text{if, } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

Clearly, $h$ is defined for every real number.

Let $c$ be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim_{x \to c} h(x) = \lim_{x \to c} [-(x+1)] = -(c+1)$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, $h$ is continuous at all points $x$, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim_{x \to c} h(x) = \lim_{x \to c} (x+1) = c+1$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, $h$ is continuous at all points $x$, such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} [-(x+1)] = -(-1+1) = 0$$

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} (x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \to -1^-} h(x) = \lim_{h \to -1^+} h(x) = h(-1)$$

Therefore, $h$ is continuous at $x = -1$

From the above three observations, it can be concluded that $h$ is continuous at all points of the real line.

$g$ and $h$ are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, $f$ has no point of discontinuity.

## Exercise-5.2

**Question 1:**

Differentiate the functions with respect to $x$.

$\sin\left(x^2+5\right)$

Let $f\left(x\right)=\sin\left(x^2+5\right)$, $u\left(x\right)=x^2+5$, and $v\left(t\right)=\sin t$

Then, $\left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f\left(x\right)$

Thus, $f$ is a composite of two functions.

Put $t=u\left(x\right)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

Therefore, by chain rule, $\dfrac{df}{dx}=\dfrac{dv}{dt}\cdot\dfrac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

**Alternate method**

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right)\cdot\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right)\cdot[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect to $x$.

$\cos(\sin x)$

Let $f\left(x\right)=\cos\left(\sin x\right), u\left(x\right)=\sin x$, and $v\left(t\right)=\cos t$

Then, $\left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)=v\left(\sin x\right)=\cos\left(\sin x\right)=f\left(x\right)$

Thus, $f$ is a composite function of two functions.

Put $t = u\ (x) = \sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$
$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

**Alternate method**

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x)\cdot\frac{d}{dx}(\sin x)=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$$

Question 3:

Differentiate the functions with respect to *x*.

$\sin(ax+b)$

Let $f(x)=\sin(ax+b)$, $u(x)=ax+b$, and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, *f* is a composite function of two functions, *u* and *v*.

Put $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$
$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

**Alternate method**

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=\cos(ax+b).(a+0)$$
$$=a\cos(ax+b)$$

**Question 4:**

Differentiate the functions with respect to *x*.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Let $f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t,$ and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, *f* is a composite function of three functions, *u, v*, and *w*.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

$$\text{Then, }\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t)\qquad[s=\tan t]$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\qquad\left[t=\sqrt{x}\right]$$
$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$
$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$
$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

**Alternate method**

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$
$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to *x*.

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g(x)=\sin(ax+b)$ and

$h(x)=\cos(cx+d)$

$$\therefore f'=\frac{g'h-gh'}{h^2}$$

Consider $g(x)=\sin(ax+b)$
Let $u(x)=ax+b, v(t)=\sin t$
Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore$ *g* is a composite function of two functions, *u* and *v*.

Put $t=u(x)=ax+b$

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$
$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Therefore, by chain rule, we obtain

$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$

Consider $h(x) = \cos(cx+d)$

Let $p(x) = cx+d,\ q(y) = \cos y$

Then, $(qop)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$

$\therefore h$ is a composite function of two functions, $p$ and $q$.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx+d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx+d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx+d) \times c = -c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b) \cdot \cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b) \cdot \frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

**Question 6:**

Differentiate the functions with respect to $x$.

$\cos x^3 . \sin^2(x^5)$

The given function is $\cos x^3 . \sin^2(x^5)$.

$$\frac{d}{dx}\left[\cos x^3 \cdot \sin^2\left(x^5\right)\right] = \sin^2\left(x^5\right) \times \frac{d}{dx}\left(\cos x^3\right) + \cos x^3 \times \frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$= \sin^2\left(x^5\right) \times \left(-\sin x^3\right) \times \frac{d}{dx}\left(x^3\right) + \cos x^3 \times 2\sin\left(x^5\right) \cdot \frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3 \sin^2\left(x^5\right) \times 3x^2 + 2\sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}\left(x^5\right)$$

$$= -3x^2 \sin x^3 \cdot \sin^2\left(x^5\right) + 2\sin x^5 \cos x^5 \cos x^3 \cdot \times 5x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2\left(x^5\right)$$

**Question 7:**

Differentiate the functions with respect to $x$.

$$2\sqrt{\cot\left(x^2\right)}$$

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$= 2 \cdot \frac{1}{2\sqrt{\cot\left(x^2\right)}} \times \frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times -\operatorname{cosec}^2\left(x^2\right) \times \frac{d}{dx}\left(x^2\right)$$

$$= -\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times \frac{1}{\sin^2\left(x^2\right)} \times (2x)$$

$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2 \cos x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sin x^2 \sqrt{\sin 2x^2}}$$

**Question 8:**

Differentiate the functions with respect to $x$.

$\cos\left(\sqrt{x}\right)$

$$\begin{aligned}
&\text{Let } f(x) = \cos\left(\sqrt{x}\right)\\
&\text{Also, let } u(x) = \sqrt{x}\\
&\text{And, } v(t) = \cos t\\
&\text{Then, } (vou)(x) = v(u(x))\\
&\qquad = v\left(\sqrt{x}\right)\\
&\qquad = \cos\sqrt{x}\\
&\qquad = f(x)
\end{aligned}$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

$$t = u(x) = \sqrt{x}$$

$$\text{Then, } \frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\text{And, } \frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t = -\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\begin{aligned}
&\frac{dt}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}\\
&= -\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}\\
&= -\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)\\
&= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}
\end{aligned}$$

**Alternate method**

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] = -\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= -\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

**Question 9:**

Prove that the function $f$ given by

$f(x) = |x-1|,\ x\in\mathbf{R}$ is notdifferentiable at $x = 1$.

The given function is $f(x) = |x-1|,\ x\in\mathbf{R}$

It is known that a function $f$ is differentiable at a point $x = c$ in its domain if both

$\lim\limits_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim\limits_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$,

consider the left hand limit of $f$ at $x = 1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^-}\frac{|h|-0}{h} = \lim_{h\to 0^-}\frac{-h}{h} \qquad (h<0\Rightarrow |h| = -h)$$

$$= -1$$

Consider the right hand limit of $f$ at $x = 1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^+}\frac{|h|-0}{h} = \lim_{h\to 0^+}\frac{h}{h} \qquad (h>0\Rightarrow |h| = h)$$

$$= 1$$

Since the left and right hand limits of $f$ at $x = 1$ are not equal, $f$ is not differentiable at $x = 1$

**Question 10:**

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$.

The given function $f$ is $f(x) = [x], 0 < x < 3$

It is known that a function $f$ is differentiable at a point $x = c$ in its domain if both

$\lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$, consider the left hand limit of $f$ at $x = 1$

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{[1+h] - [1]}{h}$$
$$= \lim_{h\to 0^-} \frac{0-1}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

Consider the right hand limit of $f$ at $x = 1$

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{[1+h] - [1]}{h}$$
$$= \lim_{h\to 0^+} \frac{1-1}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of $f$ at $x = 1$ are not equal, $f$ is not differentiable at

$x = 1$

To check the differentiability of the given function at $x = 2$, consider the left hand limit

of $f$ at $x = 2$

$$\lim_{h\to0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to0^-}\frac{[2+h]-[2]}{h}$$
$$=\lim_{h\to0^-}\frac{1-2}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of $f$ at $x=1$

$$\lim_{h\to0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to0^+}\frac{[2+h]-[2]}{h}$$
$$=\lim_{h\to0^+}\frac{2-2}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of $f$ at $x = 2$ are not equal, $f$ is not differentiable at $x = 2$

## Exercise-5.3

**Question 1:**

Find $\frac{dy}{dx}$:

$2x+3y=\sin x$

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to $x$, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$
$$\Rightarrow\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$
$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$
$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$
$$\therefore\frac{dy}{dx}=\frac{\cos x-2}{3}$$

**Question 2:**

Find $\frac{dy}{dx}$ :

$2x+3y=\sin y$

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

**Question 3:**

Find $\frac{dy}{dx}$ :

$ax+by^2=\cos y$

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ ...(2)

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$
$$\Rightarrow(2by+\sin y)\frac{dy}{dx}=-a$$
$$\therefore\frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$ :

$xy+y^2=\tan x+y$

The given relationship is $xy+y^2=\tan x+y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(xy+y^2)=\frac{d}{dx}(\tan x+y)$$
$$\Rightarrow\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$
$$\Rightarrow\left[y\cdot\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}\qquad\text{[Using product rule and chain rule]}$$
$$\Rightarrow y\cdot 1+x.\frac{dy}{dx}+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$
$$\Rightarrow(x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$
$$\therefore\frac{dy}{dx}=\frac{\sec^2 x-y}{(x+2y-1)}$$

Question 5:

Find $\frac{dy}{dx}$ :

$x^2+xy+y^2=100$

The given relationship is $x^2+xy+y^2=100$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=0$$ [Derivative of constant function is 0]

$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Question 6:

Find $\frac{dy}{dx}$ :

$$x^3+x^2y+xy^2+y^3=81$$

The given relationship is $x^3+x^2y+xy^2+y^3=81$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^3+x^2y+xy^2+y^3\right)=\frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2y\right)+\frac{d}{dx}\left(xy^2\right)+\frac{d}{dx}\left(y^3\right)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y\cdot 2x+x^2\frac{dy}{dx}\right]+\left[y^2\cdot 1+x\cdot 2y\cdot\frac{dy}{dx}\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$ :

$\sin^2 y + \cos xy = \pi$

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}\left(\cos xy\right) = 0 \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad ...(2)$$
$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$
$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$$
$$\Rightarrow \left(2\sin y\cos y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$
$$\Rightarrow \left(\sin 2y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$
$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$ :

$\sin^2 x + \cos^2 y = 1$

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 x + \cos^2 y\right) = \frac{d}{dx}(1)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right) + \frac{d}{dx}\left(\cos^2 y\right) = 0$$
$$\Rightarrow 2\sin x \cdot \frac{d}{dx}(\sin x) + 2\cos y \cdot \frac{d}{dx}(\cos y) = 0$$
$$\Rightarrow 2\sin x \cos x + 2\cos y(-\sin y) \cdot \frac{dy}{dx} = 0$$
$$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

**Question 9:**

Find $\frac{dy}{dx}$ :

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

The given relationship is $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{(1+x^2).\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2).2-2x\cdot[0+2x]}{(1+x^2)^2}=\frac{2+2x^2-4x^2}{(1+x^2)^2}=\frac{2(1-x^2)}{(1+x^2)^2} \qquad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}$$

$$=\sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}=\frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

**Question 10:**

Find $\frac{dy}{dx}$ :

$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

The given relationship is $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x - x^3}{1 - 3x^2} \qquad ...(1)$$

It is known that, $$\tan y = \frac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1 - 3\tan^2\frac{y}{3}} \qquad ...(2)$$

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3} \cdot \frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3} \cdot \frac{1}{3} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1 + \tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2}$$

**Question 11:**

Find $\frac{dy}{dx}$ :

**x** $y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), 0 < x < 1$

The given relationship is,

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2} = x$$

Differentiating this relationship with respect to *x*, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

**Question 12:**

Find $\frac{dy}{dx}$ :

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0 < x < 1$$

The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to $x$, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2 - \left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \sqrt{\frac{4x^2}{\left(1+x^2\right)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left(1+x^2\right)\cdot\left(1-x^2\right)' - \left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad \text{[Using quotient rule]}$$

$$= \frac{\left(1+x^2\right)(-2x) - \left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$= \frac{-2x - 2x^3 - 2x + 2x^3}{\left(1+x^2\right)^2}$$

$$= \frac{-4x}{\left(1+x^2\right)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx} = \frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

**Alternate method**

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y = 1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2 = 1-\sin y$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x = \frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x = \frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x = \tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to $x$, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\cdot\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]\cdot\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1=\left(1+x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

**Question 13:**

Find $\frac{dy}{dx}$ :

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$$

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\cdot\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y\cdot\frac{dy}{dx}=\frac{\left(1+x^2\right)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx} = \frac{\left(1+x^2\right)\times 2 - 2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

**Question 14:**

Find $\frac{dy}{dx}$ :

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

The given relationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$
$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to $x$, we obtain

$$\cos y \frac{dy}{dx} = 2\left[x \frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2}\cdot\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

**Question 15:**

Find $\frac{dy}{dx}$ :

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$$

The given relationship is $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2-1}$$

$$\Rightarrow \cos y = 2x^2-1$$

$$\Rightarrow 2x^2 = 1+\cos y$$

$$\Rightarrow 2x^2 = 2\cos^2\frac{y}{2}$$

$$\Rightarrow x = \cos\frac{y}{2}$$

Differentiating this relationship with respect to $x$, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1=-\sin\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin\frac{y}{2}}=\frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sin\frac{y}{2}}=\frac{-2}{\sqrt{1-\cos^2\frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$$

## Exercise-5.4

**Question 1:**

Differentiate the following w.r.t. $x$:

$$\frac{e^x}{\sin x}$$

Let $y=\frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sin x\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$=\frac{\sin x.(e^x)-e^x\cdot(\cos x)}{\sin^2 x}$$

$$=\frac{e^x(\sin x-\cos x)}{\sin^2 x},x\neq n\pi,n\in\mathbf{Z}$$

**Question 2:**

Differentiate the following w.r.t. $x$:

$e^{\sin^{-1}x}$

Let $y = e^{\sin^{-1}x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$
$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{d}{dx}\left(\sin^{-1}x\right)$$
$$= e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$$
$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$
$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

**Question 3:**

Differentiate the following w.r.t. $x$:

$e^{x^3}$

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right) = e^{x^3} \cdot \frac{d}{dx}\left(x^3\right) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

**Question 4:**

Differentiate the following w.r.t. $x$:

$\sin\left(\tan^{-1} e^{-x}\right)$

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$
$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$
$$= \cos\left(\tan^{-1} e^{-x}\right) \cdot \frac{1}{1+\left(e^{-x}\right)^2} \cdot \frac{d}{dx}\left(e^{-x}\right)$$
$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x)$$
$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$
$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

**Question 5:**

Differentiate the following w.r.t. $x$:

$\log\left(\cos e^x\right)$

Let $y = \log\left(\cos e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$
$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}\left(\cos e^x\right)$$
$$= \frac{1}{\cos e^x} \cdot \left(-\sin e^x\right) \cdot \frac{d}{dx}\left(e^x\right)$$
$$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$$
$$= -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$$

**Question 6:**

Differentiate the following w.r.t. $x$:

$e^x + e^{x^2} + \ldots + e^{x^5}$

$$\frac{d}{dx}\left(e^{x}+e^{x^2}+\ldots+e^{x^5}\right)$$
$$=\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}\left(e^{x^2}\right)+\frac{d}{dx}\left(e^{x^3}\right)+\frac{d}{dx}\left(e^{x^4}\right)+\frac{d}{dx}\left(e^{x^5}\right)$$
$$=e^{x}+\left[e^{x^2}\times\frac{d}{dx}\left(x^2\right)\right]+\left[e^{x^3}\cdot\frac{d}{dx}\left(x^3\right)\right]+\left[e^{x^4}\cdot\frac{d}{dx}\left(x^4\right)\right]+\left[e^{x^5}\cdot\frac{d}{dx}\left(x^5\right)\right]$$
$$=e^{x}+\left(e^{x^2}\times 2x\right)+\left(e^{x^3}\times 3x^2\right)+\left(e^{x^4}\times 4x^3\right)+\left(e^{x^5}\times 5x^4\right)$$
$$=e^{x}+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}$$

**Question 7:**

Differentiate the following w.r.t. $x$:

$\sqrt{e^{\sqrt{x}}}, x>0$

Let $y=\sqrt{e^{\sqrt{x}}}$

Then, $y^2=e^{\sqrt{x}}$

By differentiating this relationship with respect to $x$, we obtain

$$y^2=e^{\sqrt{x}}$$
$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$
$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x>0$$

**Question 8:**

Differentiate the following w.r.t. $x$:

$\log(\log x), x > 1$

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$
$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$
$$= \frac{1}{\log x} \cdot \frac{1}{x}$$

$$= \frac{1}{x \log x}, x > 1$$

**Question 9:**

Differentiate the following w.r.t. $x$:

$\frac{\cos x}{\log x}, x > 0$

Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$

$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2},x>0$$

Question 10:

Differentiate the following w.r.t. $x$:

$\cos\left(\log x+e^x\right),x>0$

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=-\sin\left(\log x+e^x\right)\cdot\frac{d}{dx}\left(\log x+e^x\right)$$

$$=-\sin\left(\log x+e^x\right)\cdot\left[\frac{d}{dx}(\log x)+\frac{d}{dx}\left(e^x\right)\right]$$

$$=-\sin\left(\log x+e^x\right)\cdot\left(\frac{1}{x}+e^x\right)$$

$$=-\left(\frac{1}{x}+e^x\right)\sin\left(\log x+e^x\right),x>0$$

Exercise-5.5

**Question 1:**

Differentiate the function with respect to $x$.

$\cos x.\cos 2x.\cos 3x$

Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log(\cos x.\cos 2x.\cos 3x)$$
$$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x) + \frac{1}{\cos 2x}\cdot\frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x}\cdot\frac{d}{dx}(\cos 3x)$$
$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}(3x)\right]$$
$$\therefore \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$$

**Question 2:**

Differentiate the function with respect to $x$.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm on both the sides, we obtain

$$\log y = \log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1) + \frac{1}{x-2}\cdot\frac{d}{dx}(x-2) - \frac{1}{x-3}\cdot\frac{d}{dx}(x-3) \\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4) - \frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

**Question 3:**

Differentiate the function with respect to $x$.

$$(\log x)^{\cos x}$$

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to $x$.

$x^x-2^{\sin x}$

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y=u-v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

$u=x^x$

Taking logarithm on both the sides, we obtain

$\log u=x\log x$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u}\frac{du}{dx}=\left[\frac{d}{dx}(x)\times\log x+x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log x+x\times\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^x(\log x+1)$$

$$\Rightarrow\frac{du}{dx}=x^x(1+\log x)$$

$v=2^{\sin x}$

Taking logarithm on both the sides with respect to $x$, we obtain

$$\log v = \sin x \cdot \log 2$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$
$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$$

**Question 5:**

Differentiate the function with respect to $x$.

$$(x+3)^2 . (x+4)^3 . (x+5)^4$$

Let $y = (x+3)^2 . (x+4)^3 . (x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$
$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=2\cdot\frac{1}{x+3}\cdot\frac{d}{dx}(x+3)+3\cdot\frac{1}{x+4}\cdot\frac{d}{dx}(x+4)+4\cdot\frac{1}{x+5}\cdot\frac{d}{dx}(x+5)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\cdot\left[2\left(x^2+9x+20\right)+3\left(x^2+8x+15\right)+4\left(x^2+7x+12\right)\right]$$

$$\therefore\frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\left(9x^2+70x+133\right)$$

**Question 6:**

Differentiate the function with respect to $x$.

$$\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$$

Let $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$

Also, let $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\ldots(1)$$

Then, $u=\left(x+\frac{1}{x}\right)^x$

$$\Rightarrow\log u=\log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow\log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \quad ...(2)$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right) \quad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\frac{x^{2}-1}{x^{2}+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^{2}}\right)$$

Question 7:

Differentiate the function with respect to $x$.

$$(\log x)^{x}+x^{\log x}$$

Let $y=(\log x)^{x}+x^{\log x}$

Also, let $u=(\log x)^{x}$ and $v=x^{\log x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u = (\log x)^{x}$

$$\Rightarrow \log u=\log\left[(\log x)^{x}\right]$$
$$\Rightarrow \log u=x\log(\log x)$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$
$$\Rightarrow \frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$
$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$
$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$
$$\Rightarrow \frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right] \qquad ...(2)$$

$v = x^{\log x}$

$\Rightarrow \log v = \log\left(x^{\log x}\right)$

$\Rightarrow \log v = \log x \log x = \left(\log x\right)^2$

Differentiating both sides with respect to $x$, we obtain

$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{d}{dx}\left[\left(\log x\right)^2\right]$

$\Rightarrow \frac{1}{v}.\frac{dv}{dx} = 2\left(\log x\right)\cdot\frac{d}{dx}\left(\log x\right)$

$\Rightarrow \frac{dv}{dx} = 2v\left(\log x\right)\cdot\frac{1}{x}$

$\Rightarrow \frac{dv}{dx} = 2x^{\log x}\frac{\log x}{x}$

$\Rightarrow \frac{dv}{dx} = 2x^{\log x-1}\cdot\log x \qquad ...(3)$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = \left(\log x\right)^{x-1}\left[1+\log x.\log\left(\log x\right)\right] + 2x^{\log x-1}\cdot\log x$$

**Question 8:**

Differentiate the function with respect to $x$.

$\left(\sin x\right)^x + \sin^{-1}\sqrt{x}$

Let $y = \left(\sin x\right)^x + \sin^{-1}\sqrt{x}$

Also, let $u = \left(\sin x\right)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$

$u=(\sin x)^x$

$\Rightarrow \log u=\log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to $x$, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\sin x)+x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\cdot\log(\sin x)+x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x\left[\log(\sin x)+\frac{x}{\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^x(x\cot x+\log\sin x) \qquad ...(2)$$

$v=\sin^{-1}\sqrt{x}$

Differentiating both sides with respect to $x$, we obtain

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{x-x^2}} \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\sin x)^x(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}$$

**Question 9:**

Differentiate the function with respect to $x$.

$x^{\sin x}+(\sin x)^{\cos x}$

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = x^{\sin x}$

$\Rightarrow \log u = \log\left(x^{\sin x}\right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\sin x)\cdot \log x + \sin x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\cos x \log x + \sin x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] \qquad ...(2)$$

$v = (\sin x)^{\cos x}$

$\Rightarrow \log v = \log(\sin x)^{\cos x}$

$\Rightarrow \log v = \cos x \log(\sin x)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x)\times \log(\sin x) + \cos x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = v\left[-\sin x . \log(\sin x) + \cos x \cdot \frac{1}{\sin x}\cdot \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \cot x \cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\cot x \cos x - \sin x \log \sin x\right] \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log\sin x\right]$$

**Question 10:**

Differentiate the function with respect to $x$.

$$x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

Let $y=x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$

Also, let $u=x^{x\cos x}$ and $v=\dfrac{x^2+1}{x^2-1}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{x\cos x}$

$\Rightarrow \log u=\log\left(x^{x\cos x}\right)$

$\Rightarrow \log u=x\cos x\log x$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\cdot\cos x\cdot\log x+x\cdot\frac{d}{dx}(\cos x)\cdot\log x+x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[1\cdot\cos x\cdot\log x+x\cdot(-\sin x)\log x+x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{x\cos x}\left(\cos x\log x-x\sin x\log x+\cos x\right)$$

$$\Rightarrow \frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right] \qquad ...(2)$$

$v=\dfrac{x^2+1}{x^2-1}$

$\Rightarrow \log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x\left(1+\log x\right)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

**Question 11:**

Differentiate the function with respect to $x$.

$$\left(x\cos x\right)^x+\left(x\sin x\right)^{\frac{1}{x}}$$

Let $y=\left(x\cos x\right)^x+\left(x\sin x\right)^{\frac{1}{x}}$

Also, let $u=\left(x\cos x\right)^x$ and $v=\left(x\sin x\right)^{\frac{1}{x}}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=\left(x\cos x\right)^x$

$\Rightarrow \log u=\log\left(x\cos x\right)^x$

$\Rightarrow \log u=x\log\left(x\cos x\right)$

$\Rightarrow \log u=x\left[\log x+\log\cos x\right]$

$\Rightarrow \log u=x\log x+x\log\cos x$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log x)+\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right\}+\left\{\log\cos x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[\left(\log x\cdot 1+x\cdot\frac{1}{x}\right)+\left\{\log\cos x\cdot 1+x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[(\log x+1)+\left\{\log\cos x+\frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[(1+\log x)+(\log\cos x-x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[1-x\tan x+(\log x+\log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[1-x\tan x+\log(x\cos x)\right] \qquad ...(2)$$

$$v=(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}\log x+\frac{1}{x}\log\sin x$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{x}\log x\right)+\frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}(\log x)\right]+\left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{x}\right]+\left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(1-\log x)+\left[-\frac{\log(\sin x)}{x^2}+\frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}+\frac{-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right]\qquad\ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(x\cos x)^x\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x+1-\log(x\sin x)}{x^2}\right]$$

**Question 12:**

Find $\frac{dy}{dx}$ of function.

$x^y+y^x=1$

The given function is $x^y+y^x=1$

Let $x^y=u$ and $y^x=v$

Then, the function becomes $u+v=1$

$$\therefore\frac{du}{dx}+\frac{dv}{dx}=0\qquad\ldots(1)$$

$$u = x^y$$
$$\Rightarrow \log u = \log\left(x^y\right)$$
$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad ...(2)$$

$$v = y^x$$
$$\Rightarrow \log v = \log\left(y^x\right)$$
$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$
$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) + y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$
$$\Rightarrow \left(x^y \log x + xy^{x-1}\right)\frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y\right)$$
$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

**Question 13:**

Find $\frac{dy}{dx}$ of function.

$y^x = x^y$

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$x \log y = y \log x$

Differentiating both sides with respect to $x$, we obtain

$$\log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x)$$
$$\Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x}$$
$$\Rightarrow \log y + \frac{x}{y}\frac{dy}{dx} = \log x \frac{dy}{dx} + \frac{y}{x}$$
$$\Rightarrow \left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} - \log y$$
$$\Rightarrow \left(\frac{x - y\log x}{y}\right)\frac{dy}{dx} = \frac{y - x\log y}{x}$$
$$\therefore \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

$(\cos x)^y = (\cos y)^x$

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$y \log \cos x = x \log \cos y$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$
$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$
$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$
$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$
$$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = y \tan x + \log \cos y$$
$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

**Question 15:**

Find $\frac{dy}{dx}$ of function.

$xy = e^{(x-y)}$

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$
$$\Rightarrow \log x + \log y = (x-y)\log e$$
$$\Rightarrow \log x + \log y = (x-y) \times 1$$
$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$
$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$
$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$
$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$
$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

**Question 16:**

Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

The given relationship is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}\left[f(x)\right]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$
$$\Rightarrow \frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$
$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot 4x^3+\frac{1}{1+x^8}\cdot 8x^7\right]$$
$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$
$$\text{Hence, } f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$$
$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$
$$=16\times\left(\frac{1+2+4+8}{2}\right)$$
$$=16\times\frac{15}{2}=120$$

**Question 17:**

Differentiate in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

(i)

Let $x^2-5x+8=u$ and $x^3+7x+9=v$

$$\therefore y=uv$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^2-5x+8\right)\cdot\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow \frac{dy}{dx}=(2x-5)\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+x^2\left(3x^2+7\right)-5x\left(3x^2+7\right)+8\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4+7x^2\right)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(ii)

$$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$$
$$=x^2\left(x^3+7x+9\right)-5x\left(x^3+7x+9\right)+8\left(x^3+7x+9\right)$$
$$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$
$$=x^5-5x^4+15x^3-26x^2+11x+72$$
$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^5-5x^4+15x^3-26x^2+11x+72\right)$$
$$=\frac{d}{dx}\left(x^5\right)-5\frac{d}{dx}\left(x^4\right)+15\frac{d}{dx}\left(x^3\right)-26\frac{d}{dx}\left(x^2\right)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$$
$$=5x^4-5\times4x^3+15\times3x^2-26\times2x+11\times1+0$$
$$=5x^4-20x^3+45x^2-52x+11$$

(iii) $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$
$$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$
$$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$
$$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$
$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$
$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$
$$\Rightarrow \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

**Question 18:**

If *u*, *v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v \cdot w)$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u\left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to $x$, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$
$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$
$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$
$$\Rightarrow \frac{dy}{dx} = u.v.w.\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$
$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

## Exercise-5.6

**Question 1:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = 2at^2,\ y = at^4$$

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

**Question 2:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \cos \theta$, $y = b \cos \theta$

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

**Question 3:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \sin t$, $y = \cos 2t$

The given equations are $x = \sin t$ and $y = \cos 2t$

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$$

**Question 4:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = 4t,\ y = \frac{4}{t}$$

The given equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

**Question 5:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$$

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$$

$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

**Question 6:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$$

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

**Question 7:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

The given equations are $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}\left[\dfrac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t \cdot \frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \cdot 3\sin^2 t \cdot \frac{d}{dt}\left(\sin t\right) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}} . \frac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t} \cdot \sin^2 t \cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}} . \left(-2\sin 2t\right)}{\cos 2t}$$

$$= \frac{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}{\cos 2t \sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t} . \frac{d}{dt}\left(\cos^3 t\right) - \cos^3 t \cdot \frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} . 3\cos^2 t \cdot \frac{d}{dt}\left(\cos t\right) - \cos^3 t . \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t} . \cos^2 t\left(-\sin t\right) - \cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} . \left(-2\sin 2t\right)}{\cos 2t}$$

$$= \frac{-3\cos 2t . \cos^2 t . \sin t + \cos^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t\sin 2t}{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t(2\sin t\cos t)}{3\cos 2t\sin^2 t\cos t + \sin^3 t(2\sin t\cos t)}$$

$$= \frac{\sin t\cos t\left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3(2\cos^2 t - 1)\cos t + 2\cos^3 t\right]}{\left[3(1 - 2\sin^2 t)\sin t + 2\sin^3 t\right]} \qquad \left[\begin{array}{l}\cos 2t = (2\cos^2 t - 1),\\ \cos 2t = (1 - 2\sin^2 t)\end{array}\right]$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \left[\begin{array}{l}\cos 3t = 4\cos^3 t - 3\cos t,\\ \sin 3t = 3\sin t - 4\sin^3 t\end{array}\right]$$

$$= -\cot 3t$$

**Question 8:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$$

The given equations are $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$

Then, $\frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \frac{d}{dt}(\sin t) = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a \cos t}{\left(a \frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

**Question 9:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \sec \theta,\ y = b \tan \theta$

The given equations are $x = a \sec \theta$ and $y = b \tan \theta$

Then, $\frac{dx}{d\theta} = a \cdot \frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b \cdot \frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a} \times \frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

**Question 10:**

If $x$ and $y$ are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$
$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta \cdot \frac{d}{d\theta}(\theta)\right\}\right]$$
$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$
$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

**Question 11:**

If $x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

The given equations are $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

$\Rightarrow x = \left(a^{\sin^{-1} t}\right)^{\frac{1}{2}}$ and $y = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}}$

$\Rightarrow x = a^{\frac{1}{2}\sin^{-1} t}$ and $y = a^{\frac{1}{2}\cos^{-1} t}$

Consider $x = a^{\frac{1}{2}\sin^{-1} t}$

Taking logarithm on both the sides, we obtain

$$\log x = \frac{1}{2}\sin^{-1} t \log a$$

$$\therefore \frac{1}{x}\cdot\frac{dx}{dt} = \frac{1}{2}\log a \cdot \frac{d}{dt}\left(\sin^{-1} t\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider $y = a^{\frac{1}{2}\cos^{-1} t}$

Taking logarithm on both the sides, we obtain

$$\log y = \frac{1}{2}\cos^{-1} t \log a$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\log a \cdot \frac{d}{dt}\left(\cos^{-1} t\right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved

## Exercise-5.7

**Question 1:**

Find the second order derivatives of the function.

$x^2+3x+2$

Let $y=x^2+3x+2$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$
$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

**Question 2:**

Find the second order derivatives of the function.

$x^{20}$

Let $y=x^{20}$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{20}\right)=20x^{19}$$
$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(20x^{19}\right)=20\frac{d}{dx}\left(x^{19}\right)=20\cdot 19\cdot x^{18}=380x^{18}$$

**Question 3:**

Find the second order derivatives of the function.

$x\cdot\cos x$

Let $y=x\cdot\cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot\cos x)=\cos x\cdot\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x\cdot 1+x(-\sin x)=\cos x-x\sin x$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2}&=\frac{d}{dx}[\cos x-x\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)\\&=-\sin x-\left[\sin x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\sin x)\right]\\&=-\sin x-(\sin x+x\cos x)\\&=-(x\cos x+2\sin x)\end{aligned}$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Let $y=\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3\log x$

Let $y=x^3\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x\cdot\frac{d}{dx}\left(x^3\right)+x^3\cdot\frac{d}{dx}(\log x)$$

$$=\log x\cdot 3x^2+x^3\cdot\frac{1}{x}=\log x\cdot 3x^2+x^2$$

$$=x^2(1+3\log x)$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[x^2(1+3\log x)\right]$$

$$=(1+3\log x)\cdot\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}(1+3\log x)$$

$$=(1+3\log x)\cdot 2x+x^2\cdot\frac{3}{x}$$

$$=2x+6x\log x+3x$$

$$=5x+6x\log x$$

$$=x(5+6\log x)$$

**Question 6:**

Find the second order derivatives of the function.

$e^x\sin 5x$

Let $y=e^x\sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\sin 5x\right)=\sin 5x\cdot\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}(\sin 5x)$$

$$=\sin 5x\cdot e^x+e^x\cdot\cos 5x\cdot\frac{d}{dx}(5x)=e^x\sin 5x+e^x\cos 5x\cdot 5$$

$$=e^x(\sin 5x+5\cos 5x)$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x(\sin 5x+5\cos 5x)\right]$$

$$=(\sin 5x+5\cos 5x)\cdot\frac{d}{dx}\left(e^x\right)+e^x\cdot\frac{d}{dx}(\sin 5x+5\cos 5x)$$

$$=(\sin 5x+5\cos 5x)e^x+e^x\left[\cos 5x\cdot\frac{d}{dx}(5x)+5(-\sin 5x)\cdot\frac{d}{dx}(5x)\right]$$

$$=e^x(\sin 5x+5\cos 5x)+e^x(5\cos 5x-25\sin 5x)$$

Then,

$$=e^x(10\cos 5x-24\sin 5x)=2e^x(5\cos 5x-12\sin 5x)$$

**Question 7:**

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right) = \cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\cos 3x)$$
$$= \cos 3x\cdot e^{6x}\cdot\frac{d}{dx}(6x) + e^{6x}\cdot(-\sin 3x)\cdot\frac{d}{dx}(3x)$$
$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad \ldots(1)$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$
$$= 6\cdot\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\sin 3x)\right] \qquad [\text{Using } (1)]$$
$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x\cdot e^{6x}\cdot 6 + e^{6x}\cdot\cos 3x\cdot 3\right]$$
$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$
$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$
$$= 9e^{6x}(3\cos 3x - 4\sin 3x)$$

**Question 8:**

Find the second order derivatives of the function.

$\tan^{-1} x$

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(1+x^2\right)^{-1} = (-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$
$$= \frac{-1}{\left(1+x^2\right)^2}\times 2x = \frac{-2x}{\left(1+x^2\right)^2}$$

**Question 9:**

Find the second order derivatives of the function.

$\log(\log x)$

Let $y = \log(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right] = \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) = \frac{1}{x \log x} = (x \log x)^{-1}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}\left[(x \log x)^{-1}\right] = (-1) \cdot (x \log x)^{-2} \cdot \frac{d}{dx}(x \log x)$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)\right]$$

$$= \frac{-1}{(x \log x)^2} . \left[\log x \cdot 1 + x \cdot \frac{1}{x}\right] = \frac{-(1 + \log x)}{(x \log x)^2}$$

**Question 10:**

Find the second order derivatives of the function.

$\sin(\log x)$

Let $y = \sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\log x)\right]=\cos(\log x)\cdot\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$=\frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right]-\cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right]-\cos(\log x).1}{x^2}$$

$$=\frac{-x\sin(\log x).\frac{1}{x}-\cos(\log x)}{x^2}$$

$$=\frac{-\left[\sin(\log x)+\cos(\log x)\right]}{x^2}$$

**Question 11:**

If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

It is given that, $y=5\cos x-3\sin x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)$$

$$=5(-\sin x)-3\cos x=-(5\sin x+3\cos x)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(5\sin x+3\cos x)\right]$$

$$=-\left[5\cdot\frac{d}{dx}(\sin x)+3\cdot\frac{d}{dx}(\cos x)\right]$$

$$=-\left[5\cos x+3(-\sin x)\right]$$

$$=-\left[5\cos x-3\sin x\right]$$

$$=-y$$

$$\therefore \frac{d^2y}{dx^2}+y=0$$

Hence, proved.

**Question 12:**

If $y=\cos^{-1}x,$ find $\frac{d^2y}{dx^2}$ in terms of $y$ alone.

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}=-(1-x^2)^{\frac{-1}{2}}$$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(1-x^2)^{\frac{-1}{2}}\right]$$
$$=-\left(-\frac{1}{2}\right)\cdot(1-x^2)^{\frac{-3}{2}}\cdot\frac{d}{dx}(1-x^2)$$
$$=\frac{1}{2\sqrt{(1-x^2)^3}}\times(-2x)$$
$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{(1-x^2)^3}}\qquad\ldots(\text{i})$$
$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}}$$
$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{(\sin^2 y)^3}}$$
$$=\frac{-\cos y}{\sin^3 y}$$
$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$
$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

**Question 13:**

If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

It is given that, $y=3\cos(\log x)+4\sin(\log x)$

Then,

$$y_1 = 3\cdot\frac{d}{dx}\left[\cos(\log x)\right] + 4\cdot\frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] + 4\cdot\left[\cos(\log x)\cdot\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x\left[4\{\cos(\log x)\}' - 3\{\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

**Question 14:**

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A\cdot\frac{d}{dx}\left(e^{mx}\right) + B\cdot\frac{d}{dx}\left(e^{nx}\right) = A\cdot e^{mx}.\frac{d}{dx}(mx) + B\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am\cdot\frac{d}{dx}\left(e^{mx}\right) + Bn\cdot\frac{d}{dx}\left(e^{nx}\right)$$

$$= Am\cdot e^{mx}\cdot\frac{d}{dx}(mx) + Bn\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

**Question 15:**

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)$$

$$= 500\cdot e^{7x}\cdot\frac{d}{dx}(7x) + 600\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$

$$= 3500e^{7x} - 4200e^{-7x}$$

$$\therefore \frac{d^2y}{dx^2} = 3500\cdot\frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$

$$= 3500\cdot e^{7x}.\frac{d}{dx}(7x) - 4200\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$

$$= 7\times3500\cdot e^{7x} + 7\times4200\cdot e^{-7x}$$

$$= 49\times500e^{7x} + 49\times600e^{-7x}$$

$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$

$$= 49y$$

Hence, proved.

**Question 16:**

If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

The given relationship is $e^y(x+1)=1$

$$e^y(x+1)=1$$
$$\Rightarrow e^y=\frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$
$$\therefore\frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$
$$\Rightarrow\frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$
$$\Rightarrow\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

**Question 17:**

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2y_2+2x\left(x^2+1\right)y_1=2$

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$y_1 = 2\tan^{-1} x \frac{d}{dx}\left(\tan^{-1} x\right)$

$\Rightarrow y_1 = 2\tan^{-1} x . \frac{1}{1+x^2}$

$\Rightarrow \left(1+x^2\right) y_1 = 2\tan^{-1} x$

Again differentiating with respect to $x$ on both the sides, we obtain

$\left(1+x^2\right) y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$

$\Rightarrow \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right) y_1 = 2$

Hence, proved.

## Exercise-5.8

**Question 1:**

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8\ , x \in [-4, 2]$

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$f(-4) = (-4)^2 + 2\times(-4) - 8 = 16 - 8 - 8 = 0$

$f(2) = (2)^2 + 2\times 2 - 8 = 4 + 4 - 8 = 0$

$\therefore f(-4) = f(2) = 0$

$\Rightarrow$ The value of $f(x)$ at −4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$f(x) = x^2 + 2x - 8$

$\Rightarrow f'(x) = 2x + 2$

$\therefore f'(c) = 0$

$\Rightarrow 2c + 2 = 0$

$\Rightarrow c = -1,\ \text{where } c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified for the given function.

**Question 2:**

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x \in [5, 9]$

(ii) $f(x)=[x]$ for $x \in [-2, 2]$

(iii) $f(x)=x^2-1$ for $x \in [1, 2]$

By Rolle's Theorem, for a function $f:[a, b] \to \mathbf{R}$, if

(a) $f$ is continuous on $[a, b]$

(b) $f$ is differentiable on $(a, b)$

(c) $f(a) = f(b)$

then, there exists some $c \in (a, b)$ such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$.

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$
$\therefore f(5) \neq f(9)$

The differentiability of $f$ in $(5, 9)$ is checked as follows.

Let $n$ be an integer such that $n \in (5, 9)$.

The left hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of $f$ at $x = n$ are not equal, $f$ is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that $f$ does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of $f$ in $(-2, 2)$ is checked as follows.

Let $n$ be an integer such that $n \in (-2, 2)$.

The left hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-}\frac{[n+h]-[n]}{h} = \lim_{h\to 0^-}\frac{n-1-n}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+}\frac{[n+h]-[n]}{h} = \lim_{h\to 0^+}\frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of $f$ at $x = n$ are not equal, $f$ is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that $f$ does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that $f$, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$$f(1) = (1)^2 - 1 = 0$$
$$f(2) = (2)^2 - 1 = 3$$

$\therefore f(1) \neq f(2)$

It is observed that $f$ does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Question 3:

If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

It is given that $f:[-5,5]\rightarrow \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) *f* is continuous on [−5, 5].

(b) *f* is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists *c* ∈ (−5, 5) such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$
$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c)\neq 0$$
$$\Rightarrow 10f'(c)\neq 0$$
$$\Rightarrow f(5)-f(-5)\neq 0$$
$$\Rightarrow f(5)\neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval $[a,\ b]$, where $a=1$ and $b=4$.

The given function is $f(x)=x^2-4x-3$

*f*, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is 2*x* − 4.

$$f(1)=1^2-4\times1-3=-6,\ f(4)=4^2-4\times4-3=-3$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c)=1$

$$f'(c)=1$$
$$\Rightarrow 2c-4=1$$
$$\Rightarrow c=\frac{5}{2}, \text{ where } c=\frac{5}{2}\in(1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

**Question 5:**

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c\in(1,3)$ for which $f'(c)=0$

The given function $f$ is $f(x)=x^3-5x^2-3x$

$f$, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1)=1^3-5\times1^2-3\times1=-7,\ f(3)=3^3-5\times3^2-3\times3=-27$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c)=-10$

$$f'(c)=-10$$
$$\Rightarrow 3c^2-10c-3=10$$
$$\Rightarrow 3c^2-10c+7=0$$
$$\Rightarrow 3c^2-3c-7c+7=0$$
$$\Rightarrow 3c(c-1)-7(c-1)=0$$
$$\Rightarrow (c-1)(3c-7)=0$$
$$\Rightarrow c=1,\ \frac{7}{3}, \text{ where } c=\frac{7}{3}\in(1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c=\frac{7}{3}\in(1,\ 3)$ is the only point for which $f'(c)=0$

**Question 6:**

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Mean Value Theorem states that for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) $f$ is continuous on $[a, b]$

(b) $f$ is differentiable on $(a, b)$

then, there exists some $c \in (a, b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in $[5, 9]$.

The differentiability of $f$ in $(5, 9)$ is checked as follows.

Let $n$ be an integer such that $n \in (5, 9)$.

The left hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of $f$ at $x = n$ are not equal, $f$ is not differentiable at $x = n$

∴ $f$ is not differentiable in (5, 9).

It is observed that $f$ does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

⇒ $f(x)$ is not continuous in $[-2, 2]$.

The differentiability of $f$ in $(-2, 2)$ is checked as follows.

Let $n$ be an integer such that $n \in (-2, 2)$.

The left hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of $f$ at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of $f$ at $x = n$ are not equal, $f$ is not differentiable at $x = n$

$\therefore f$ is not differentiable in (−2, 2).

It is observed that $f$ does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that $f$, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that $f$ satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

It can be proved as follows.

$$f(1)=1^2-1=0,\ f(2)=2^2-1=3$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$$

$$f'(x)=2x$$
$$\therefore f'(c)=3$$
$$\Rightarrow 2c=3$$
$$\Rightarrow c=\frac{3}{2}=1.5,\ \text{where } 1.5\in[1,\ 2]$$