Exercise -7.1

Question 1:

sin 2x

The anti derivative of sin 2x is a function of x whose derivative is sin 2x. It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$
$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$
$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$

Question 2:

Cos 3x

The anti derivative of cos 3x is a function of x whose derivative is cos 3x. It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$
$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$
$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

Question 3:

e^{2x}

The anti derivative of e^{2x} is the function of x whose derivative is e^{2x}. It is known that,

$$\frac{d}{dx}\left(e^{2x}\right)=2e^{2x}$$
$$\Rightarrow e^{2x}=\frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$
$$\therefore e^{2x}=\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

$(ax+b)^2$

The anti derivative of $(ax+b)^2$ is the function of x whose derivative is $(ax+b)^2$.

It is known that,

$$\frac{d}{dx}(ax+b)^3=3a(ax+b)^2$$
$$\Rightarrow (ax+b)^2=\frac{1}{3a}\frac{d}{dx}(ax+b)^3$$
$$\therefore (ax+b)^2=\frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Question 5:

$\sin 2x-4e^{3x}$

The anti derivative of $\left(\sin 2x-4e^{3x}\right)$ is the function of x whose derivative is $\left(\sin 2x-4e^{3x}\right)$.

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)=\sin 2x-4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x-4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)$.

Question 6:

$$\int\left(4e^{3x}+1\right)dx$$

$$\int\left(4e^{3x}+1\right)dx$$

$$=4\int e^{3x}dx+\int 1dx$$

$$=4\left(\frac{e^{3x}}{3}\right)+x+C$$

$$=\frac{4}{3}e^{3x}+x+C$$

Question 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$=\int\left(x^2-1\right)dx$$

$$=\int x^2dx-\int 1dx$$

$$=\frac{x^3}{3}-x+C$$

Question 8:

$$\int\left(ax^2+bx+c\right)dx$$

$$\int\left(ax^2+bx+c\right)dx$$

$$= a\int x^2 dx + b\int x dx + c\int 1.dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

Question 9:

$$\int\left(2x^2 + e^x\right)dx$$

$$\int\left(2x^2 + e^x\right)dx$$

$$= 2\int x^2 dx + \int e^x dx$$

$$= 2\left(\frac{x^3}{3}\right) + e^x + C$$

$$= \frac{2}{3}x^3 + e^x + C$$

Question 10:

$$\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

$$\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int\left(x + \frac{1}{x} - 2\right)dx$$

$$= \int x dx + \int\frac{1}{x}dx - 2\int 1.dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

Question 11:

$$\int\frac{x^3 + 5x^2 - 4}{x^2}dx$$

$\int \frac{x^3+5x^2-4}{x^2}dx$

$= \int \left(x+5-4x^{-2}\right)dx$

$= \int x dx + 5\int 1.dx - 4\int x^{-2}dx$

$= \frac{x^2}{2}+5x-4\left(\frac{x^{-1}}{-1}\right)+C$

$= \frac{x^2}{2}+5x+\frac{4}{x}+C$

Question 12:

$\int \frac{x^3+3x+4}{\sqrt{x}}dx$

$\int \frac{x^3+3x+4}{\sqrt{x}}dx$

$= \int \left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$

$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$

$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$

$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$

Question 13:

$\int \frac{x^3-x^2+x-1}{x-1}dx$

$\int \frac{x^3-x^2+x-1}{x-1}dx$

On dividing, we obtain

$$= \int (x^2+1)dx$$
$$= \int x^2 dx + \int 1 dx$$
$$= \frac{x^3}{3} + x + C$$

Question 14:

$$\int (1-x)\sqrt{x}dx$$

$$\int (1-x)\sqrt{x}dx$$

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}}\right)dx$$
$$= \int x^{\frac{1}{2}}dx - \int x^{\frac{3}{2}}dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$
$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

Question 15:

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx$$

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx$$

$$= \int \left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)dx$$
$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$
$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + 3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + C$$
$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

Question 16:

$\int\left(2x-3\cos x+e^{x}\right)dx$

$\int\left(2x-3\cos x+e^{x}\right)dx$

$=2\int x dx-3\int\cos x dx+\int e^{x}dx$

$=\frac{2x^{2}}{2}-3(\sin x)+e^{x}+C$

$=x^{2}-3\sin x+e^{x}+C$

Question 17:

$\int\left(2x^{2}-3\sin x+5\sqrt{x}\right)dx$

$\int\left(2x^{2}-3\sin x+5\sqrt{x}\right)dx$

$=2\int x^{2}dx-3\int\sin x dx+5\int x^{\frac{1}{2}}dx$

$=\frac{2x^{3}}{3}-3(-\cos x)+5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$

$=\frac{2}{3}x^{3}+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$

Question 18:

$\int\sec x(\sec x+\tan x)dx$

$\int\sec x(\sec x+\tan x)dx$

$=\int\left(\sec^{2}x+\sec x\tan x\right)dx$

$=\int\sec^{2}x dx+\int\sec x\tan x dx$

$=\tan x+\sec x+C$

Question 19:

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$$

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \tan^2 x dx$$

$$= \int (\sec^2 x - 1) dx$$

$$= \int \sec^2 x dx - \int 1 dx$$

$$= \tan x - x + C$$

Question 20:

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$$

$$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$$

$$= 2\tan x - 3\sec x + C$$

Question 21:

The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$ (D) $\frac{3}{2}x^{\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}+C$

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

$$=\int x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C$$

$$=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$$

Hence, the correct answer is C.

Question 22:

If $\frac{d}{dx}f(x)=4x^{3}-\frac{3}{x^{4}}$ such that f(2) = 0, then f(x) is

(A) $x^{4}+\frac{1}{x^{3}}-\frac{129}{8}$ (B) $x^{3}+\frac{1}{x^{4}}+\frac{129}{8}$

(C) $x^{4}+\frac{1}{x^{3}}+\frac{129}{8}$ (D) $x^{3}+\frac{1}{x^{4}}-\frac{129}{8}$

It is given that,

$\frac{d}{dx}f(x)=4x^{3}-\frac{3}{x^{4}}$

∴Anti derivative of

$$4x^{3}-\frac{3}{x^{4}}=f(x)$$

$$\therefore f(x)=\int 4x^3-\frac{3}{x^4}dx$$

$$f(x)=4\int x^3dx-3\int(x^{-4})dx$$

∴

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

Also,

$$f(x)=x^4+\frac{1}{x^3}+C$$

$$f(2)=0$$

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

Hence, the correct answer is A.

Exercise -7.2

Question 1:

$$\frac{2x}{1+x^2}$$

Let $1+x^2=t$

$\therefore 2x\ dx=dt$

$$\Rightarrow \int\frac{2x}{1+x^2}dx=\int\frac{1}{t}dt$$

$= \log|t| + C$

$= \log\left|1 + x^2\right| + C$

$= \log\left(1 + x^2\right) + C$

Question 2:

$\dfrac{(\log x)^2}{x}$

Let log |x| = t

∴

1

$$\Rightarrow \int \frac{(\log|x|)^2}{x}\,dx = \int t^2\,dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

Question 3:

$\dfrac{1}{x + x\log x}$

$$\frac{1}{x + x\log x} = \frac{1}{x(1 + \log x)}$$

Let 1 + log x = t

∴

1

$$\Rightarrow \int \frac{1}{x(1 + \log x)}\,dx = \int \frac{1}{t}\,dt$$

$= \log|t| + C$

$= \log|1 + \log x| + C$

Question 4:

sin x · sin (cos x) sin x ·
sin (cos x) Let cos x = t
∴ −sin x dx = dt

$$\Rightarrow \int \sin x \cdot \sin(\cos x)\, dx = -\int \sin t\, dt$$
$$= -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

Question 5:

$$\sin(ax+b)\cos(ax+b)$$

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b) = t$
∴ 2adx = dt

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2} dx = \frac{1}{2}\int \frac{\sin t\ dt}{2a}$$
$$= \frac{1}{4a}[-\cos t] + C$$
$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

Question 6:

$$\sqrt{ax+b}$$

Let ax + b = t
⇒ adx = dt

$$\therefore dx = \frac{1}{a} dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}} dx = \frac{1}{a} \int t^{\frac{1}{2}} dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 7:

$$x\sqrt{x+2}$$

Let $(x+2) = t$

$\therefore dx = dt$

$$\Rightarrow \int x\sqrt{x+2}dx = \int (t-2)\sqrt{t}dt$$

$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) dt$$

$$= \int t^{\frac{3}{2}} dt - 2\int t^{\frac{1}{2}} dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8:

$$x\sqrt{1+2x^2}$$

Let $1 + 2x^2 = t$

$\therefore 4x\,dx = dt$

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}\,dt}{4}$$
$$= \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$
$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

Question 9:

$$(4x+2)\sqrt{x^2+x+1}$$

Let $x^2+x+1=t$

$\therefore (2x+1)dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$= \int 2\sqrt{t}\,dt$$
$$= 2\int \sqrt{t}\,dt$$
$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

Question 10:

$$\frac{1}{x-\sqrt{x}}$$

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right)=t$

∴

$$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx = \int \frac{2}{t}\,dt$$

$$= 2\log|t| + C$$

$$= 2\log\left|\sqrt{x}-1\right| + C$$

Question 11:

$$\frac{x}{\sqrt{x+4}}, x > 0$$

Let $x+4=t$

∴ dx = dt

$$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$$

$$= \int\left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$$

$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}\,(x-8) + C$$

Question 12:

$\left(x^3-1\right)^{\frac{1}{3}}x^5$

Let $x^3-1=t$

$\therefore 3x^2dx=dt$

$$\Rightarrow \int\left(x^3-1\right)^{\frac{1}{3}}x^5dx=\int\left(x^3-1\right)^{\frac{1}{3}}x^3\cdot x^2dx$$
$$=\int t^{\frac{1}{3}}(t+1)\frac{dt}{3}$$
$$=\frac{1}{3}\int\left(t^{\frac{4}{3}}+t^{\frac{1}{3}}\right)dt$$
$$=\frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}}+\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right]+C$$
$$=\frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}}+\frac{3}{4}t^{\frac{4}{3}}\right]+C$$
$$=\frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}}+\frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}}+C$$

Question 13:

$\frac{x^2}{\left(2+3x^3\right)^3}$

Let $2+3x^3=t$

$\therefore 9x^2\,dx=dt$

$$\Rightarrow \int\frac{x^2}{\left(2+3x^3\right)^3}dx=\frac{1}{9}\int\frac{dt}{(t)^3}$$
$$=\frac{1}{9}\left[\frac{t^{-2}}{-2}\right]+C$$
$$=\frac{-1}{18}\left(\frac{1}{t^2}\right)+C$$
$$=\frac{-1}{18\left(2+3x^3\right)^2}+C$$

Question 14:

$$\frac{1}{x(\log x)^m}, x > 0$$

Let log x = t

∴ $\frac{1}{x}dx = dt$

$$\Rightarrow \int \frac{1}{x(\log x)^m} dx = \int \frac{dt}{(t)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15:

$$\frac{x}{9-4x^2}$$

Let $9 - 4x^2 = t$

∴ −8x dx = dt

$$\Rightarrow \int \frac{x}{9-4x^2} dx = \frac{-1}{8} \int \frac{1}{t} dt$$

$$= \frac{-1}{8} \log|t| + C$$

$$= \frac{-1}{8} \log|9-4x^2| + C$$

Question 16:

$$e^{2x+3}$$

Let $2x + 3 = t$

∴ 2dx = dt

$$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t dt$$
$$= \frac{1}{2}\left(e^t\right) + C$$
$$= \frac{1}{2}e^{(2x+3)} + C$$

Question 17:

$$\frac{x}{e^{x^2}}$$

Let $x^2 = t$
$\therefore$ 2xdx = dt

$$\Rightarrow \int \frac{x}{e^{x^2}}dx = \frac{1}{2}\int \frac{1}{e^t}dt$$

$$= \frac{1}{2}\int e^{-t}dt$$
$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$$
$$= -\frac{1}{2}e^{-x^2} + C$$
$$= \frac{-1}{2e^{x^2}} + C$$

Question 18:

$$\frac{e^{\tan^{-1}x}}{1+x^2}$$

Let $\tan^{-1}x = t$
$\therefore$

1

$$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt$$

$$= e^t + C$$
$$= e^{\tan^{-1}x} + C$$

Question 19:

$$\frac{e^{2x}-1}{e^{2x}+1}$$

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let $e^x+e^{-x}=t$

$\therefore \left(e^x-e^{-x}\right)dx=dt$

$$\Rightarrow \int\frac{e^{2x}-1}{e^{2x}+1}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$=\int\frac{dt}{t}$$
$$=\log|t|+C$$
$$=\log\left|e^x+e^{-x}\right|+C$$

Question 20:

$$\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$$

Let $e^{2x}+e^{-2x}=t$

$\therefore \left(2e^{2x}-2e^{-2x}\right)dx=dt$

$$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx = \int\frac{dt}{2t}$$
$$= \frac{1}{2}\int\frac{1}{t}\,dt$$
$$= \frac{1}{2}\log|t| + C$$
$$= \frac{1}{2}\log\left|e^{2x}+e^{-2x}\right| + C$$

Question 21:

$$\tan^2(2x-3)$$

$$\tan^2(2x-3) = \sec^2(2x-3) - 1$$

Let 2x − 3 = t
∴ 2dx = dt

$$\Rightarrow \int\tan^2(2x-3)\,dx = \int\left[\left(\sec^2(2x-3)\right)-1\right]dx$$
$$= \frac{1}{2}\int\left(\sec^2 t\right)dt - \int 1\,dx$$
$$= \frac{1}{2}\int\sec^2 t\,dt - \int 1\,dx$$
$$= \frac{1}{2}\tan t - x + C$$
$$= \frac{1}{2}\tan(2x-3) - x + C$$

Question 22:

$$\sec^2(7-4x)$$

Let 7 − 4x = t
∴ −4dx = dt

$$\therefore \int\sec^2(7-4x)\,dx = \frac{-1}{4}\int\sec^2 t\,dt$$
$$= \frac{-1}{4}(\tan t) + C$$
$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 23:

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

Let $\sin^{-1} x = t$

∴

1

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int t\, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{\left(\sin^{-1} x\right)^2}{2} + C$$

Question 24:

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2\left(3\cos x + 2\sin x\right)}$$

Let $3\cos x + 2\sin x = t$

∴ $\left(-3\sin x + 2\cos x\right) dx = dt$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\, dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}\, dt$$

$$= \frac{1}{2}\log\left|t\right| + C$$

$$= \frac{1}{2}\log\left|2\sin x + 3\cos x\right| + C$$

Question 25:

$$\frac{1}{\cos^2 x(1-\tan x)^2}$$

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x) = t$

$\therefore -\sec^2 x dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{(1-\tan x)^2} dx = \int \frac{-dt}{t^2}$$
$$= -\int t^{-2} dt$$
$$= +\frac{1}{t} + C$$
$$= \frac{1}{(1-\tan x)} + C$$

Question 26:

$$\frac{\cos\sqrt{x}}{\sqrt{x}}$$

Let $\sqrt{x} = t$

$\therefore$

1

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t \, dt$$
$$= 2\sin t + C$$
$$= 2\sin\sqrt{x} + C$$

Question 27:

$$\sqrt{\sin 2x}\cos 2x$$

Let sin 2x = t

∴

$2\cos 2x\,dx = dt$

$$\Rightarrow \int \sqrt{\sin 2x}\ \cos 2x\,dx = \frac{1}{2}\int \sqrt{t}\,dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3}t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

Question 28:

$$\frac{\cos x}{\sqrt{1+\sin x}}$$

Let $1+\sin x = t$

∴ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}}\,dx = \int \frac{dt}{\sqrt{t}}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1+\sin x} + C$$

Question 29:

cot x log sin x

Let log sin x = t

$\Rightarrow \frac{1}{\sin x} \cdot \cos x \; dx = dt$

$\therefore \cot x \; dx = dt$

$$\Rightarrow \int \cot x \log \sin x \; dx = \int t \, dt$$
$$= \frac{t^2}{2} + C$$
$$= \frac{1}{2}(\log \sin x)^2 + C$$

Question 30:

$\frac{\sin x}{1 + \cos x}$

Let 1 + cos x = t

$\therefore -\sin x \, dx = dt$

$$\Rightarrow \int \frac{\sin x}{1 + \cos x} dx = \int -\frac{dt}{t}$$
$$= -\log|t| + C$$
$$= -\log|1 + \cos x| + C$$

Question 31:

$\frac{\sin x}{(1 + \cos x)^2}$

Let 1 + cos x = t

$\therefore -\sin x \, dx = dt$

$$\Rightarrow \int \frac{\sin x}{(1 + \cos x)^2} dx = \int -\frac{dt}{t^2}$$
$$= -\int t^{-2} dt$$
$$= \frac{1}{t} + C$$
$$= \frac{1}{1 + \cos x} + C$$

Question 32:

$$\frac{1}{1+\cot x}$$

$$\text{Let } I = \int \frac{1}{1+\cot x} dx$$
$$= \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$$
$$= \int \frac{\sin x}{\sin x + \cos x} dx$$
$$= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x} dx$$
$$= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$$
$$= \frac{1}{2} \int 1\, dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

Let sin x + cos x = t ⇒ (cos x − sin x) dx = dt

$$= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

$$\therefore I = \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t}$$
$$= \frac{x}{2} - \frac{1}{2} \log|t| + C$$
$$= \frac{x}{2} - \frac{1}{2} \log|\sin x + \cos x| + C$$

Question 33:

$$\frac{1}{1-\tan x}$$

$$\text{Let } I = \int \frac{1}{1-\tan x}dx$$
$$= \int \frac{1}{1-\frac{\sin x}{\cos x}}dx$$
$$= \int \frac{\cos x}{\cos x - \sin x}dx$$
$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}dx$$
$$= \frac{1}{2}\int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)}dx$$

Put cos x − sin x = t ⇒ (−sin x − cos x) dx = dt

$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}dx$$

$$\frac{+\sin x}{-\sin x}dx$$

$$\therefore I = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$$
$$= \frac{x}{2} - \frac{1}{2}\log|t| + C$$
$$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$

$$\text{Let } I = \int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$$
$$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x}dx$$
$$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x}dx$$
$$= \int \frac{\sec^2 x\,dx}{\sqrt{\tan x}}$$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{t}}$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{\tan x} + C$$

Question 35:

$$\frac{(1+\log x)^2}{x}$$

Let 1 + log x = t

∴

$$\Rightarrow \int \frac{(1+\log x)^2}{x}\,dx = \int t^2\,dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{(1+\log x)^3}{3} + C$$

Question 36:

$$\frac{(x+1)(x+\log x)^2}{x}$$

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x) = t$

∴

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx = \int t^2\,dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{1}{3}(x+\log x)^3 + C$$

Question 37:

$$\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$$

Let $x^4 = t$

$\therefore 4x^3\,dx = dt$

$$\Rightarrow \int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8}\,dx = \frac{1}{4}\int \frac{\sin\left(\tan^{-1} t\right)}{1+t^2}\,dt \qquad ...(1)$$

Let $\tan^{-1} t = u$

$\therefore$

$\frac{1}{1+t^2}\,dt = du$

From (1), we obtain

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)dx}{1+x^8} = \frac{1}{4}\int \sin u\,du$$

$$= \frac{1}{4}\left(-\cos u\right) + C$$

$$= \frac{-1}{4}\cos\left(\tan^{-1} t\right) + C$$

$$= \frac{-1}{4}\cos\left(\tan^{-1} x^4\right) + C$$

Question 38:

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $\left(10^x - x^{10}\right)^{-1} + C$ (D) $\log\left(10^x + x^{10}\right) + C$

Let $x^{10} + 10^x = t$

$\therefore \left(10x^9 + 10^x \log_e 10\right)dx = dt$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right) + C$$

Hence, the correct answer is D.

Question 39:

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

A. $\tan x + \cot x + C$

B. $\tan x - \cot x + C$

C. $\tan x \cot x + C$

D. $\tan x - \cot 2x + C$

$$\begin{aligned}
\text{Let } I &= \int \frac{dx}{\sin^2 x \cos^2 x} \\
&= \int \frac{1}{\sin^2 x \cos^2 x} dx \\
&= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\
&= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\
&= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx \\
&= \tan x - \cot x + C
\end{aligned}$$

Hence, the correct answer is B.

Exercise -7.3

Question 1:

$\sin^2(2x+5)$

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)\,dx = \int \frac{1-\cos(4x+10)}{2}dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(4x+10)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$

$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

Question 2:

$\sin 3x \cos 4x$

It is known that, $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$\therefore \int \sin 3x \cos 4x\,dx = \frac{1}{2}\int \{\sin(3x+4x) + \sin(3x-4x)\}\,dx$$

$$= \frac{1}{2}\int \{\sin 7x + \sin(-x)\}\,dx$$

$$= \frac{1}{2}\int \{\sin 7x - \sin x\}\,dx$$

$$= \frac{1}{2}\int \sin 7x\,dx - \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$$

Question 3:

cos 2x cos 4x cos 6x

It is known that, $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

$$\therefore \int \cos 2x(\cos 4x \cos 6x)\,dx = \int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right]dx$$

$$= \frac{1}{2}\int\{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}\,dx$$

$$= \frac{1}{2}\int\{\cos 2x \cos 10x + \cos^2 2x\}\,dx$$

$$= \frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$$

$$= \frac{1}{4}\int(\cos 12x + \cos 8x + 1 + \cos 4x)\,dx$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+C$$

Question 4:

$\sin^3(2x+1)$

Let $I = \int \sin^3(2x+1)$

$$\Rightarrow \int \sin^3(2x+1)\,dx = \int \sin^2(2x+1)\cdot\sin(2x+1)\,dx$$

$$= \int(1-\cos^2(2x+1))\sin(2x+1)\,dx$$

Let $\cos(2x+1) = t$

$\Rightarrow -2\sin(2x+1)\,dx = dt$

$\Rightarrow \sin(2x+1)\,dx = \frac{-dt}{2}$

$$\Rightarrow I = \frac{-1}{2}\int(1-t^2)\,dt$$

$$= \frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}$$

$$= \frac{-1}{2}\left\{\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right\}$$

$$= \frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C$$

Question 5:

$\sin^3 x \cos^3 x$

$$\text{Let } I = \int \sin^3 x \cos^3 x \cdot dx$$
$$= \int \cos^3 x \cdot \sin^2 x \cdot \sin x \cdot dx$$
$$= \int \cos^3 x \left(1 - \cos^2 x\right) \sin x \cdot dx$$

$$\text{Let } \cos x = t$$
$$\Rightarrow -\sin x \cdot dx = dt$$
$$\Rightarrow I = -\int t^3 \left(1 - t^2\right) dt$$
$$= -\int \left(t^3 - t^5\right) dt$$
$$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$
$$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C$$
$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

Question 6:

$\sin x \sin 2x \sin 3x$

It is known that, $\sin A \sin B = \frac{1}{2}\left\{\cos\left(A - B\right) - \cos\left(A + B\right)\right\}$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \int \left[\sin x \cdot \frac{1}{2} \{\cos(2x-3x) - \cos(2x+3x)\} \right] dx$$
$$= \frac{1}{2} \int (\sin x \cos(-x) - \sin x \cos 5x) \, dx$$
$$= \frac{1}{2} \int (\sin x \cos x - \sin x \cos 5x) \, dx$$
$$= \frac{1}{2} \int \frac{\sin 2x}{2} dx - \frac{1}{2} \int \sin x \cos 5x \, dx$$
$$= \frac{1}{4} \left[\frac{-\cos 2x}{2} \right] - \frac{1}{2} \int \left\{ \frac{1}{2} \sin(x+5x) + \sin(x-5x) \right\} dx$$
$$= \frac{-\cos 2x}{8} - \frac{1}{4} \int (\sin 6x + \sin(-4x)) \, dx$$
$$= \frac{-\cos 2x}{8} - \frac{1}{4} \left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4} \right] + C$$
$$= \frac{-\cos 2x}{8} - \frac{1}{8} \left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2} \right] + C$$
$$= \frac{1}{8} \left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x \right] + C$$

Question 7:

sin 4x sin 8x

It is known that, $\sin A \sin B = \frac{1}{2} \cos(A-B) - \cos(A+B)$

$$\therefore \int \sin 4x \sin 8x \, dx = \int \left\{ \frac{1}{2} \cos(4x-8x) - \cos(4x+8x) \right\} dx$$
$$= \frac{1}{2} \int (\cos(-4x) - \cos 12x) \, dx$$
$$= \frac{1}{2} \int (\cos 4x - \cos 12x) \, dx$$
$$= \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right]$$

Question 8:

$$\frac{1-\cos x}{1+\cos x}$$

$$\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[2\sin^2\frac{x}{2}=1-\cos x \text{ and } 2\cos^2\frac{x}{2}=1+\cos x\right]$$

$$=\tan^2\frac{x}{2}$$

$$=\left(\sec^2\frac{x}{2}-1\right)$$

$$\therefore \int\frac{1-\cos x}{1+\cos x}dx=\int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$=\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}-x\right]+C$$

$$=2\tan\frac{x}{2}-x+C$$

Question 9:

$$\frac{\cos x}{1+\cos x}$$

$$\frac{\cos x}{1+\cos x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int\frac{\cos x}{1+\cos x}dx=\frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$=\frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$=x-\tan\frac{x}{2}+C$$

Question 10:

$\sin^4 x$

$$\sin^4 x = \sin^2 x \sin^2 x$$
$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{4}(1-\cos 2x)^2$$
$$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$
$$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$\therefore \int \sin^4 x\, dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right) - \frac{2\sin 2x}{2}\right] + C$$
$$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4} - 2\sin 2x\right] + C$$
$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Question 11:

$\cos^4 2x$

$$\cos^4 2x = (\cos^2 2x)^2$$
$$= \left(\frac{1+\cos 4x}{2}\right)^2$$
$$= \frac{1}{4}\left[1+\cos^2 4x + 2\cos 4x\right]$$
$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right) + 2\cos 4x\right]$$
$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2} + 2\cos 4x\right]$$
$$= \frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2} + 2\cos 4x\right]$$
$$\therefore \int \cos^4 2x\, dx = \int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$$
$$= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

Question 12:

$$\frac{\sin^2 x}{1+\cos x}$$

$$\frac{\sin^2 x}{1+\cos x}=\frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \quad \left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2};\ \cos x = 2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$=2\sin^2\frac{x}{2}$$

$$=1-\cos x$$

$$\therefore \int\frac{\sin^2 x}{1+\cos x}dx = \int(1-\cos x)dx$$

$$= x-\sin x + C$$

Question 13:

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$$

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \quad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$=2\left[\cos(x)+\cos\alpha\right]$$

$$=2\cos x+2\cos\alpha$$

$$\therefore \int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=\int 2\cos x+2\cos\alpha$$

$$=2\left[\sin x+x\cos\alpha\right]+C$$

Question 14:

$$\frac{\cos x-\sin x}{1+\sin 2x}$$

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x}$$

$$\left[\sin^2 x+\cos^2 x=1;\ \sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{(\sin x+\cos x)^2}$$

Let $\sin x+\cos x=t$

$\therefore (\cos x-\sin x)\,dx=dt$

$$\Rightarrow \int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{(\sin x+\cos x)^2}dx$$

$$=\int\frac{dt}{t^2}$$

$$=\int t^{-2}dt$$

$$=-t^{-1}+C$$

$$=-\frac{1}{t}+C$$

$$=\frac{-1}{\sin x+\cos x}+C$$

Question 15:

$$\tan^3 2x\sec 2x$$

$$\tan^3 2x\sec 2x=\tan^2 2x\tan 2x\sec 2x$$

$$=\left(\sec^2 2x-1\right)\tan 2x\sec 2x$$

$$=\sec^2 2x\cdot\tan 2x\sec 2x-\tan 2x\sec 2x$$

$$\therefore \int\tan^3 2x\sec 2x\ dx=\int\sec^2 2x\tan 2x\sec 2x\ dx-\int\tan 2x\sec 2x\ dx$$

$$=\int\sec^2 2x\tan 2x\sec 2x\ dx-\frac{\sec 2x}{2}+C$$

Let $\sec 2x=t$

$\therefore 2\sec 2x\tan 2x\ dx=dt$

$$\therefore \int\tan^3 2x\sec 2x\ dx=\frac{1}{2}\int t^2dt-\frac{\sec 2x}{2}+C$$

$$=\frac{t^3}{6}-\frac{\sec 2x}{2}+C$$

$$=\frac{(\sec 2x)^3}{6}-\frac{\sec 2x}{2}+C$$

Question 16:

$\tan^4 x$

$$\begin{aligned}
&\tan^4 x \\
&= \tan^2 x \cdot \tan^2 x \\
&= \left(\sec^2 x - 1\right)\tan^2 x \\
&= \sec^2 x \tan^2 x - \tan^2 x \\
&= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right) \\
&= \sec^2 x \tan^2 x - \sec^2 x + 1
\end{aligned}$$

$$\begin{aligned}
\therefore \int \tan^4 x\, dx &= \int \sec^2 x \tan^2 x\, dx - \int \sec^2 x\, dx + \int 1 \cdot dx \\
&= \int \sec^2 x \tan^2 x\, dx - \tan x + x + C \qquad \ldots(1)
\end{aligned}$$

Consider $\int \sec^2 x \tan^2 x\, dx$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Question 17:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

$$\begin{aligned}
\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} &= \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \\
&= \tan x \sec x + \cot x \operatorname{cosec} x
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\, dx &= \int \left(\tan x \sec x + \cot x \operatorname{cosec} x\right) dx \\
&= \sec x - \operatorname{cosec} x + C
\end{aligned}$$

Question 18:

$$\frac{\cos 2x+2\sin^2 x}{\cos^2 x}$$

$$\frac{\cos 2x+2\sin^2 x}{\cos^2 x}$$
$$=\frac{\cos 2x+(1-\cos 2x)}{\cos^2 x} \qquad \left[\cos 2x=1-2\sin^2 x\right]$$
$$=\frac{1}{\cos^2 x}$$
$$=\sec^2 x$$
$$\therefore \int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}\,dx=\int\sec^2 x\,dx=\tan x+C$$

Question 19:

$$\frac{1}{\sin x\cos^3 x}$$

$$\frac{1}{\sin x\cos^3 x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}$$
$$=\frac{\sin x}{\cos^3 x}+\frac{1}{\sin x\cos x}$$
$$=\tan x\sec^2 x+\frac{\frac{1\cos^2 x}{\sin x\cos x}}{\cos^2 x}$$
$$=\tan x\sec^2 x+\frac{\sec^2 x}{\tan x}$$

$$\therefore \int\frac{1}{\sin x\cos^3 x}dx=\int\tan x\sec^2 x\,dx+\int\frac{\sec^2 x}{\tan x}\,dx$$
Let $\tan x=t\Rightarrow \sec^2 x\,dx=dt$
$$\Rightarrow \int\frac{1}{\sin x\cos^3 x}dx=\int t\,dt+\int\frac{1}{t}\,dt$$
$$=\frac{t^2}{2}+\log|t|+C$$
$$=\frac{1}{2}\tan^2 x+\log|\tan x|+C$$

Question 20:

$\frac{\cos 2x}{(\cos x+\sin x)^2}$

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$

$$\therefore \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\int\frac{\cos 2x}{(1+\sin 2x)}\,dx$$

Let $1+\sin 2x=t$

$\Rightarrow 2\cos 2x\,dx=dt$

$$\therefore \int\frac{\cos 2x}{(\cos x+\sin x)^2}\,dx=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|1+\sin 2x|+C$$

$$=\frac{1}{2}\log\left|(\sin x+\cos x)^2\right|+C$$

$$=\log|\sin x+\cos x|+C$$

Question 21:

$\sin^{-1}(\cos x)$

$\sin^{-1}(\cos x)$

Let $\cos x=t$

Then, $\sin x=\sqrt{1-t^2}$

$$\Rightarrow (-\sin x)\,dx = dt$$
$$dx = \frac{-dt}{\sin x}$$
$$dx = \frac{-dt}{\sqrt{1-t^2}}$$
$$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$
$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}dt$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}}dt = du$$
$$\therefore \int \sin^{-1}(\cos x)dx = \int -u\,du$$
$$= -\frac{u^2}{2} + C$$
$$= \frac{-(\sin^{-1} t)^2}{2} + C$$
$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \quad \ldots(1)$$

It is known that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$
$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)\,dx = \frac{-\left[\frac{\pi}{2} - x\right]^2}{2} + C$$
$$= -\frac{1}{2}\left(\frac{\pi^2}{2} + x^2 - \pi x\right) + C$$
$$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$$
$$= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right)$$
$$= \frac{\pi x}{2} - \frac{x^2}{2} + C_1$$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

$$\frac{1}{\cos(x-a)\cos(x-b)}=\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$
$$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$
$$=\frac{1}{\sin(a-b)}\frac{[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)}$$
$$=\frac{1}{\sin(a-b)}[\tan(x-b)-\tan(x-a)]$$

$$\Rightarrow\int\frac{1}{\cos(x-a)\cos(x-b)}dx=\frac{1}{\sin(a-b)}\int[\tan(x-b)-\tan(x-a)]dx$$
$$=\frac{1}{\sin(a-b)}[-\log|\cos(x-b)|+\log|\cos(x-a)|]$$
$$=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Question 23:

$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ is equal to

A. tan x + cot x + C

B. tan x + cosec x + C

C. − tan x + cot x + C

D. tan x + sec x + C

$$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx=\int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$
$$=\int(\sec^2 x-\operatorname{cosec}^2 x)\,dx$$
$$=\tan x+\cot x+C$$

Hence, the correct answer is A.

Question 24:

$\int\frac{e^x(1+x)}{\cos^2(e^x x)}dx$ equals

A. − cot (ex^x) + C

B. tan (xe^x) + C

C. tan (e^x) + C

D. cot (e^x) + C

$$\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$$

Let $ex^x = t$

$$\Rightarrow (e^x \cdot x + e^x \cdot 1) dx = dt$$
$$e^x(x+1) dx = dt$$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \int \frac{dt}{\cos^2 t}$$
$$= \int \sec^2 t \, dt$$
$$= \tan t + C$$
$$= \tan(e^x \cdot x) + C$$

Hence, the correct answer is B.

Exercise -7.4

Question 1:

$$\frac{3x^2}{x^6+1}$$

Let $x^3 = t$

$\therefore 3x^2 \, dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Let 2x = t
∴ 2dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$
$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$
$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Let 2 − x = t
⇒ −dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$
$$= -\log\left|t+\sqrt{t^2+1}\right| + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$
$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right| + C$$
$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Let 5x = t
∴ 5dx = dt

$$\Rightarrow \int\frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int\frac{1}{9-t^2}\,dt$$
$$= \frac{1}{5}\int\frac{1}{\sqrt{3^2-t^2}}\,dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right)+C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Let $\sqrt{2}x^2 = t$
$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int\frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int\frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1}t\right]+C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right)+C$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Let $x^3 = t$
∴ $3x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6} dx = \frac{1}{3} \int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \quad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}} dx$, let $x^2 - 1 = t \Rightarrow 2x\, dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2} \int t^{-\frac{1}{2}} dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}} dt = \log\left|x + \sqrt{x^2-a^2}\right|\right]$$
$$= \sqrt{x^2-1} - \log\left|x + \sqrt{x^2-1}\right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Let $x^3 = t$

$\Rightarrow 3x^2\,dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right|+C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right|+C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Let tan x = t

$\therefore \sec^2 x\,dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$

$$= \log\left|t+\sqrt{t^2+4}\right|+C$$

$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$$

Question 10:

$$\frac{1}{\sqrt{x^2+2x+2}}$$

$$\int \frac{1}{\sqrt{x^2+2x+2}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}dx$$

Let $x+1=t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{t^2+1}}dt$$

$$= \log\left|t+\sqrt{t^2+1}\right|+C$$

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+C$$

$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

Question 11:

$$\frac{1}{\sqrt{9x^2+6x+5}}$$

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1)=t$

$\therefore\ 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$.

Therefore,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore\ \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

$$x^2-3x+2$$

$$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx=\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\Rightarrow\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$$\Rightarrow\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Let $4x+1=A\frac{d}{dx}\left(2x^2+x-3\right)+B$

$\Rightarrow 4x+1=A(4x+1)+B$

$\Rightarrow 4x+1=4Ax+A+B$

Equating the coefficients of x and constant term on both sides, we obtain

$4A=4\Rightarrow A=1$

$A+B=1\Rightarrow B=0$

Let $2x^2 + x - 3 = t$
$\therefore (4x + 1)\,dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2+x-3} + C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad ...(1)$$
$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$$(x+2) = \frac{1}{2}(2x)+2$$
$$\text{Then, } \int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$
$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$
$$\text{In } \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx, \text{ let } x^2-1 = t \;\Rightarrow\; 2x\,dx = dt$$
$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\left[2\sqrt{t}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$
$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

$$\text{Let } 5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$
$$\Rightarrow 5x-2 = A(2+6x)+B$$

Equating the coefficient of x and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$
$$2A+B = -2 \Rightarrow B = -\frac{11}{3}$$
$$\therefore 5x-2 = \frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$$
$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$
$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$
$$\text{Let } I_1 = \int \frac{2+6x}{1+2x+3x^2}dx \text{ and } I_2 = \int \frac{1}{1+2x+3x^2}dx$$
$$\therefore \int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

$$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$$
$$\text{Let } 1+2x+3x^2 = t$$
$$\Rightarrow (2+6x)dx = dt$$
$$\therefore I_1 = \int \frac{dt}{t}$$
$$I_1 = \log|t|$$
$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

$$I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$
$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$
$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$
$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$
$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$
$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$
$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$
$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$
$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right]-\frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$
$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}=\frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7=A\frac{d}{dx}\left(x^2-9x+20\right)+B$$
$$\Rightarrow 6x+7=A(2x-9)+B$$

Equating the coefficients of x and constant term, we obtain 2A = 6 ⇒ A = 3
−9A + B = 7 ⇒ B = 34
∴ 6x + 7 = 3 (2x − 9) + 34

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}=\int\frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$
$$=3\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx+34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\text{Let } I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx \text{ and } I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\therefore \int\frac{6x+7}{\sqrt{x^2-9x+20}}=3I_1+34I_2 \qquad \ldots(1)$$

Then,

$$I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2-9x+20=t$

$$\Rightarrow (2x-9)dx=dt$$

$$\Rightarrow I_1=\frac{dt}{\sqrt{t}}$$

$$I_1=2\sqrt{t}$$

$$I_1=2\sqrt{x^2-9x+20} \qquad \ldots(2)$$

$$\text{and } I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

$$\text{Let } x+2=A\frac{d}{dx}\left(4x-x^2\right)+B$$

$$\Rightarrow x+2=A(4-2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A = 1 \Rightarrow A = -\frac{1}{2}$

$4A + B = 2 \Rightarrow B = 4$

$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \quad ...(1)$$

Then. $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$\Rightarrow (4-2x)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \quad ...(2)$$

$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\Rightarrow 4x - x^2 = -(-4x + x^2)$$
$$= (-4x + x^2 + 4 - 4)$$
$$= 4 - (x-2)^2$$
$$= (2)^2 - (x-2)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$
$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x+2)\,dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \quad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$
$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$$
$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Question 22:

$$\frac{x+3}{x^2-2x-5}$$

$$\text{Let } (x+3) = A\frac{d}{dx}\left(x^2-2x-5\right)+B$$

$$(x+3) = A(2x-2)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2)+4$$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

$$\text{Let } I_1 = \int \frac{2x-2}{x^2-2x-5}dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$\therefore \int \frac{x+3}{\left(x^2-2x-5\right)}dx = \frac{1}{2}I_1 + 4I_2 \quad \ldots(1)$$

$$\text{Then, } I_1 = \int \frac{2x-2}{x^2-2x-5}dx$$

Let $x^2-2x-5=t$

$\Rightarrow (2x-2)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right| \qquad ...(2)$$

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$= \int \frac{1}{(x^2-2x+1)-6}dx$$

$$= \int \frac{1}{(x-1)^2+(\sqrt{6})^2}dx$$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \qquad ...(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Let $5x+3 = A\frac{d}{dx}(x^2+4x+10)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 5 \Rightarrow A = \frac{5}{2}$

$4A + B = 3 \Rightarrow B = -7$

$\therefore 5x + 3 = \frac{5}{2}(2x+4) - 7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2 + 4x + 10 = t$

$\therefore (2x+4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}}dx$$

$$= \int \frac{1}{(x+2)^2+(\sqrt{6})^2}dx$$

$$= \log\left|(x+2)\sqrt{x^2+4x+10}\right| \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Question 24:

$\int \frac{dx}{x^2+2x+2}$ equals

A. $x \tan^{-1}(x+1) + C$

B. $\tan^{-1}(x+1) + C$

C. $(x+1)\tan^{-1} x + C$

D. $\tan^{-1} x + C$

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{\left(x^2+2x+1\right)+1}$$
$$= \int \frac{1}{(x+1)^2+(1)^2} dx$$
$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct answer is B.

Question 25:

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

B. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

D. $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$= \int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a}+C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct answer is B.

Exercise -7.5

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$$\Rightarrow x = A(x+2)+B(x+1)$$

Equating the coefficients of x and constant term, we obtain

A + B = 1

2A + B = 0

On solving, we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

Question 2:

$$\frac{1}{x^2-9}$$

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constant term, we obtain

A + B = 0

−3A + 3B = 1

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\Rightarrow \int \frac{1}{(x^2-9)} dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx$$

$$= -\frac{1}{6} \log|x+3| + \frac{1}{6} \log|x-3| + C$$

$$= \frac{1}{6} \log \left| \frac{(x-3)}{(x+3)} \right| + C$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \qquad ...(1)$$

Substituting x = 1, 2, and 3 respectively in equation (1), we obtain

A = 1, B = −5, and C = 4

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Let $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \qquad ...(1)$$

Substituting x = 1, 2, and 3 respectively in equation (1), we obtain $A = \frac{1}{2}, B = -2, \text{ and } C = \frac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx$$

$$= \frac{1}{2} \log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

Question 5:

$$\frac{2x}{x^2+3x+2}$$

Let $\frac{2x}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$$2x = A(x+2) + B(x+1) \qquad \text{...(1)}$$

Substituting x = −1 and −2 in equation (1), we obtain

A = −2 and B = 4

$$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx$$

$$= 4\log|x+2| - 2\log|x+1| + C$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

Logic How to Make factor

$$\frac{(1 - x^2)}{x(1 - 2x)}$$ divide and multiply by 2

$$\frac{2(1 - x^2)}{2x(1 - 2x)}$$ now open the bracket $$\frac{2 - 2x^2}{2x(1 - 2x)}$$

Now add and minus the variable X

$$\frac{2 - 2x^2 + x - x}{2x(1 - 2x)} \rightarrow \frac{(x - 2x^2) + (2 - x)}{2x(1 - 2x)} \rightarrow \frac{x(1 - 2x) + (2 - x)}{2x(1 - 2x)}$$

Now divide separately

$$\frac{1}{2} + \frac{1(2 - x)}{2x(1 - 2x)}$$

To Be Continued Further

Let

$$\Rightarrow (2-x) = A(1-2x) + Bx \qquad ...(1)$$

Substituting x = 0 and $\frac{1}{2}$ in equation (1), we obtain

A = 2 and B = 3

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)}dx = \int \left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{1-2x}\right)\right\}dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0$

$-A + B = 1$

$-B + C = 0$

On solving these equations, we obtain

$$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{(x^2+1)(x-1)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Consider $\int\frac{2x}{x^2+1}dx$, let $(x^2+1)=t \Rightarrow 2x\,dx=dt$

$$\Rightarrow \int\frac{2x}{x^2+1}dx=\int\frac{dt}{t}=\log|t|=\log|x^2+1|$$

$$\therefore \int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

$$=\frac{1}{2}\log|x-1|-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+C$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Let $\frac{x}{(x-1)^2(x+2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+2)}$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Substituting x = 1, we obtain

$B = \frac{1}{3}$

Equating the coefficients of x^2 and constant term, we obtain

A + C = 0

−2A + 2B + C = 0 On

solving, we obtain

$A = \frac{2}{9}$ and $C = \frac{-2}{9}$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)} dx = \frac{2}{9}\int \frac{1}{(x-1)} dx + \frac{1}{3}\int \frac{1}{(x-1)^2} dx - \frac{2}{9}\int \frac{1}{(x+2)} dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

Question 9:

$$\frac{3x+5}{x^3 - x^2 - x + 1}$$

$$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$3x+5 = A(x^2-1) + B(x+1) + C(x^2+1-2x) \qquad ...(1)$$

Substituting x = 1 in equation (1), we obtain

B = 4

Equating the coefficients of x^2 and x, we obtain

A + C = 0

B − 2C = 3

On solving, we obtain

$$A = -\frac{1}{2} \text{ and } C = \frac{1}{2}$$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int \frac{1}{x-1}dx + 4\int \frac{1}{(x-1)^2}dx + \frac{1}{2}\int \frac{1}{(x+1)}dx$$

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$
$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$
$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2 and x, we obtain

$$B = -\frac{1}{10},\ A = \frac{5}{2},\ \text{and } C = -\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$
$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$
$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$
$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \qquad ...(1)$$

Substituting x = −1, −2, and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3},\ B = -\frac{5}{2},\ \text{and } C = \frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3}\int \frac{1}{(x+1)} dx - \frac{5}{2}\int \frac{1}{(x+2)} dx + \frac{5}{6}\int \frac{1}{(x-2)} dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Question 12:

$$\frac{x^3+x+1}{x^2-1}$$

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain

$$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$$2x+1 = A(x-1) + B(x+1) \qquad ...(1)$$

Substituting x = 1 and −1 in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\Rightarrow \int \frac{x^3+x+1}{x^2-1} dx = \int x\,dx + \frac{1}{2}\int \frac{1}{(x+1)} dx + \frac{3}{2}\int \frac{1}{(x-1)} dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Question 13:

$$\frac{2}{(1-x)(1+x^2)}$$

$$\text{Let } \frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain

A − B = 0

B − C = 0

A + C = 2

On solving these equations, we obtain

A = 1, B = 1, and C = 1

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$$

Question 14:

$$\frac{3x-1}{(x+2)^2}$$

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x - 1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we obtain

A = 3
2A + B = −1 ⇒ B = −7

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{x}{(x+2)^2}dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Question 15:

$$\frac{1}{x^4-1}$$

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$$A+B+C=0$$
$$-A+B+D=0$$
$$A+B-C=0$$
$$-A+B-D=1$$

On solving these equations, we obtain

$A=-\frac{1}{4}, B=\frac{1}{4}, C=0,$ and $D=-\frac{1}{2}$

$$\therefore \frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$$

$$\Rightarrow \int \frac{1}{x^4-1}dx=-\frac{1}{4}\log|x-1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

Question 16:

$\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x(x^n+1)}=\frac{x^{n-1}}{x^{n-1}x(x^n+1)}=\frac{x^{n-1}}{x^n(x^n+1)}$$

Let $x^n=t \Rightarrow x^{n-1}dx=dt$

$$\therefore \int \frac{1}{x(x^n+1)}dx=\int \frac{x^{n-1}}{x^n(x^n+1)}dx=\frac{1}{n}\int \frac{1}{t(t+1)}dt$$

Let $\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$

$1=A(1+t)+Bt \qquad ...(1)$

Substituting t = 0, −1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x\left(x^n+1\right)}dx = \frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(t+1)}\right\}dx$$

$$= \frac{1}{n}\left[\log|t| - \log|t+1|\right] + C$$

$$= -\frac{1}{n}\left[\log\left|x^n\right| - \log\left|x^n+1\right|\right] + C$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$$

Question 17:

$\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint: Put sin x = t]

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$

$1 = A(2-t) + B(1-t) \qquad \ldots(1)$

Substituting t = 2 and then t = 1 in equation (1), we obtain

A = 1 and B = −1

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\} dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Question 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$$\text{Let } \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 4

4A + 3C = 0

4B + 3D = 10

On solving these equations, we obtain

A = 0, B = −2, C = 0, and D = 6

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)}=\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1-\left(\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

$$=\int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}$$

$$=x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int\frac{2x}{(x^2+1)(x^2+3)}dx=\int\frac{dt}{(t+1)(t+3)} \qquad ...(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)}=\frac{A}{(t+1)}+\frac{B}{(t+3)}$$

$$1=A(t+3)+B(t+1) \qquad ...(1)$$

Substituting t = −3 and t = −1 in equation (1), we obtain

$$A=\frac{1}{2} \text{ and } B=-\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)}dx = \int\left\{\frac{1}{2(t+1)} - \frac{1}{2(t+3)}\right\}dt$$

$$= \frac{1}{2}\log|(t+1)| - \frac{1}{2}\log|t+3| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

Question 20:

$$\frac{1}{x(x^4-1)}$$

$$\frac{1}{x(x^4-1)}$$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

Let $x^4 = t \Rightarrow 4x^3dx = dt$

$$\therefore \int \frac{1}{x(x^4-1)}dx = \int \frac{x^3}{x^4(x^4-1)}dx$$

$$\therefore \int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$$

$$1 = A(t-1) + Bt \qquad ...(1)$$

Substituting t = 0 and 1 in (1), we obtain

A = −1 and B = 1

$$\Rightarrow \frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int\left\{\frac{-1}{t} + \frac{1}{t-1}\right\}dt$$

$$= \frac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$$

Question 21:

$\frac{1}{(e^x-1)}$ [Hint: Put $e^x = t$]

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{1}{e^x-1}dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)}dt$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

$$1 = A(t-1) + Bt \qquad \ldots(1)$$

Substituting t = 1 and t = 0 in equation (1), we obtain

A = −1 and B = 1

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$
$$= \log\left|\frac{e^x-1}{e^x}\right| + C$$

Question 22:

$\int \frac{xdx}{(x-1)(x-2)}$ equals

A. $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

B. $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D. $\log\left|(x-1)(x-2)\right| + C$

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1)$ ...(1)

Substituting x = 1 and 2 in (1), we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$
$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int\left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\} dx$$
$$= -\log|x-1| + 2\log|x-2| + C$$
$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct answer is B.

Question 23:

$\int \frac{dx}{x\left(x^2+1\right)}$ equals

A. $\log|x| - \frac{1}{2}\log\left(x^2+1\right) + C$

B. $\log|x| + \frac{1}{2}\log\left(x^2+1\right) + C$

C. $-\log|x| + \frac{1}{2}\log\left(x^2+1\right) + C$

D. $\frac{1}{2}\log|x| + \log\left(x^2+1\right) + C$

$$\text{Let } \frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A\left(x^2+1\right) + \left(Bx+C\right)x$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

C = 0

A = 1

On solving these equations, we obtain

A = 1, B = −1, and C = 0

$$\therefore \frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$$

$$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$$

Hence, the correct answer is A.

Exercise -7.6

Question 1:

x sin x

Let I = $\int x\sin x\,dx$

Taking x as first function and sin x as second function and integrating by parts, we obtain

$$I=x\int \sin x\,dx-\int\left\{\left(\frac{d}{dx}x\right)\int \sin x\,dx\right\}dx$$

$$=x(-\cos x)-\int 1\cdot(-\cos x)dx$$

$$=-x\cos x+\sin x+C$$

Question 2:

$x\sin 3x$

Let I = $\int x\sin 3x\,dx$

Taking x as first function and sin 3x as second function and integrating by parts, we obtain

$$I = x\int \sin 3x\, dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x\, dx\right\}$$
$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\, dx$$
$$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

Question 3:

$x^2 e^x$

Let $I = \int x^2 e^x dx$

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$I = x^2\int e^x\, dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$
$$= x^2 e^x - \int 2x\cdot e^x dx$$
$$= x^2 e^x - 2\int x\cdot e^x dx$$

Again integrating by parts, we obtain

$$= x^2 e^x - 2\left[x\cdot\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\cdot\int e^x dx\right\}dx\right]$$
$$= x^2 e^x - 2\left[xe^x - \int e^x dx\right]$$
$$= x^2 e^x - 2\left[xe^x - e^x\right]$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$
$$= e^x\left(x^2 - 2x + 2\right) + C$$

Question 4:

x logx

Let $I = \int x\log x dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$$I = \log x \int x\, dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x\, dx \right\} dx$$
$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$
$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

Question 5:

x log 2x

Let $I = \int x \log 2x dx$

Taking log 2x as first function and x as second function and integrating by parts, we obtain

$$I = \log 2x \int x\, dx - \int \left\{ \left(\frac{d}{dx} 2\log x \right) \int x\, dx \right\} dx$$
$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

Question 6:

x^2 log x

Let $I = \int x^2 \log x\, dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2\,dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x^2 dx\right\}dx$$

$$= \log x\left(\frac{x^3}{3}\right) - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3\log x}{3} - \int \frac{x^2}{3}\,dx$$

$$= \frac{x^3\log x}{3} - \frac{x^3}{9} + \text{C}$$

Question 7:

$x\sin^{-1}x$

Let $I = \int x\sin^{-1}x\;dx$

Taking $\sin^{-1}x$ as first function and x as second function and integrating by parts, we obtain

$$I = \sin^{-1}x\int x\;dx - \int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int x\;dx\right\}dx$$

$$= \sin^{-1}x\left(\frac{x^2}{2}\right) - \int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\int\left\{\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\left\{\int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}\,dx\right\}$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - \sin^{-1}x\right\} + \text{C}$$

$$= \frac{x^2\sin^{-1}x}{2} + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + \text{C}$$

$$= \frac{1}{4}\left(2x^2-1\right)\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + \text{C}$$

Question 8:

$x\tan^{-1}x$

Let $I=\int x\tan^{-1}x\,dx$

Taking $\tan^{-1}x$ as first function and x as second function and integrating by parts, we obtain

$$I=\tan^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int x\,dx\right\}dx$$
$$=\tan^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$
$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$
$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(\frac{x^2+1}{1+x^2}-\frac{1}{1+x^2}\right)dx$$
$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$
$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\left(x-\tan^{-1}x\right)+C$$
$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$$

Question 9:

$x\cos^{-1}x$

Let $I=\int x\cos^{-1}x\,dx$

Taking $\cos^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I=\cos^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$$

$$=\cos^{-1}x\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}I_1-\frac{1}{2}\cos^{-1}x \qquad \ldots(1)$$

where, $I_1=\int\sqrt{1-x^2}dx$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{d}{dx}\sqrt{1-x^2}\int x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-2x}{2\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{\int\sqrt{1-x^2}\,dx+\int\frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1=x\sqrt{1-x^2}-\left\{I_1+\cos^{-1}x\right\}$$

$$\Rightarrow 2I_1=x\sqrt{1-x^2}-\cos^{-1}x$$

$$\therefore I_1=\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x$$

Substituting in (1), we obtain

$$I=\frac{x\cos^{-1}x}{2}-\frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\cos^{-1}x\right)-\frac{1}{2}\cos^{-1}x$$

$$=\frac{(2x^2-1)}{4}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$$

Question 10:

$(\sin^{-1}x)^2$

Let $I=\int(\sin^{-1}x)^2\cdot 1\,dx$

Taking $\left(\sin^{-1}x\right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I=\left(\sin^{-1}x\right)\int 1\,dx-\int\left\{\frac{d}{dx}\left(\sin^{-1}x\right)^2\cdot\int 1\cdot dx\right\}dx$$
$$=\left(\sin^{-1}x\right)^2\cdot x-\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\cdot x\,dx$$
$$=x\left(\sin^{-1}x\right)^2+\int\sin^{-1}x\cdot\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$
$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}\,dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\cdot 2\sqrt{1-x^2}-\int\frac{1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$
$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2\,dx$$
$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+\text{C}$$

Question 11:

$$\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$$

Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}\cdot\cos^{-1}x\,dx$$

Taking $\cos^{-1}x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$=\frac{-1}{2}\left[\cos^{-1}x\cdot2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot2\sqrt{1-x^2}\,dx\right]$$
$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int2\,dx\right]$$
$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$$
$$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

Question 12:

$x\sec^2 x$

Let $I=\int x\sec^2 x dx$

Taking x as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$I=x\int\sec^2 x\,dx-\int\left\{\left\{\frac{d}{dx}x\right\}\int\sec^2 x\,dx\right\}dx$$
$$=x\tan x-\int1\cdot\tan x dx$$
$$=x\tan x+\log\left|\cos x\right|+C$$

Question 13:

$\tan^{-1}x$

Let $I=\int1\cdot\tan^{-1}x dx$

Taking $\tan^{-1}x$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int 1 dx - \int\left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int 1\cdot dx\right\}dx$$
$$= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2}\cdot x\, dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$
$$= x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right| + C$$
$$= x\tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right) + C$$

Question 14:

$x(\log x)^2$

$$I = \int x(\log x)^2\, dx$$

Taking $(\log x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = (\log x)^2 \int x\, dx - \int\left[\left\{\left(\frac{d}{dx}\log x\right)^2\right\}\int x\, dx\right]dx$$
$$= \frac{x^2}{2}(\log x)^2 - \left[\int 2\log x\cdot\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$
$$= \frac{x^2}{2}(\log x)^2 - \int x\log x\, dx$$

Again integrating by parts, we obtain

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x\int x\, dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x\, dx\right\}dx\right]$$
$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2} - \log x - \int\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$
$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\, dx$$
$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

Question 15:

$\left(x^2+1\right)\log x$

Let $I = \int (x^2+1)\log x\,dx = \int x^2 \log x\,dx + \int \log x\,dx$

Let I = $I_1 + I_2$ … (1)

Where, $I_1 = \int x^2 \log x\,dx$ and $I_2 = \int \log x\,dx$

$$I_1 = \int x^2 \log x dx$$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$\begin{aligned}
I_1 &= \log x - \int x^2 dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx \\
&= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \\
&= \frac{x^3}{3} \log x - \frac{1}{3} \left(\int x^2 dx \right) \\
&= \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 \qquad \text{... (2)}
\end{aligned}$$

$$I_2 = \int \log x\,dx$$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$\begin{aligned}
I_2 &= \log x \int 1 \cdot dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int 1 \cdot dx \right\} \\
&= \log x \cdot x - \int \frac{1}{x} \cdot x dx \\
&= x \log x - \int 1 dx \\
&= x \log x - x + C_2 \qquad \text{... (3)}
\end{aligned}$$

Using equations (2) and (3) in (1), we obtain

$$I = \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$$
$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2)$$
$$= \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$$

Question 16:

$e^x(\sin x + \cos x)$

Let $I = \int e^x(\sin x + \cos x)\,dx$

Let $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\therefore I = \int e^x\{f(x) + f'(x)\}\,dx$

It is known that, $\int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

Question 17:

$\dfrac{xe^x}{(1+x)^2}$

Let $I = \int \frac{xe^x}{(1+x)^2}dx = \int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$

$$= \int e^x\left\{\frac{1+x-1}{(1+x)^2}\right\}dx$$
$$= \int e^x\left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\}dx$$

Let $\Rightarrow$

$f(x)=\frac{1}{1+x}$ $\quad f'(x)=\frac{-1}{(1+x)^2}$

$$\Rightarrow \int \frac{xe^x}{(1+x)^2}\,dx=\int e^x\left\{f(x)+f'(x)\right\}dx$$

It is known that, $\int e^x\left\{f(x)+f'(x)\right\}dx=e^x f(x)+\mathrm{C}$

$$\therefore \int \frac{xe^x}{(1+x)^2}\,dx=\frac{e^x}{1+x}+\mathrm{C}$$

Question 18:

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$
$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$
$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$
$$=\frac{1}{2}e^x\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$
$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$
$$=\frac{1}{2}e^2\left(1+\tan\frac{x}{2}\right)^2$$
$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$
$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$
$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \qquad \ldots(1)$$

Let $\tan\frac{x}{2} = f(x)$ ⇒ $f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int e^x\left\{f(x)+f'(x)\right\}dx = e^x f(x)+C$

From equation (1), we obtain

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx = e^x\tan\frac{x}{2}+C$$

Question 19:

$$e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$$

Also, let $I = \int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$

It is known that, $\int e^x\left\{f(x)+f'(x)\right\}dx = e^x f(x)+C$

Let $\frac{1}{x} = f(x)$ ⇒ $f'(x) = -\frac{1}{x^2}$

$$\therefore I = \frac{e^x}{x}+C$$

Question 20:

$$\frac{(x-3)e^x}{(x-1)^3}$$

$$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx = \int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$$

$$= \int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$$

Let $f(x) = \frac{1}{(x-1)^2}$ ⇒ $f'(x) = \frac{-2}{(x-1)^3}$

It is known that, $\int e^x \{f(x)+f'(x)\} dx = e^x f(x)+C$

$$\therefore \int e^x \left\{\frac{(x-3)}{(x-1)^2}\right\} dx = \frac{e^x}{(x-1)^2}+C$$

Question 21:

$e^{2x} \sin x$

Let $I = \int e^{2x} \sin x \, dx \qquad ...(1)$

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2}\left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2}\left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5}\left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5}\left[2 \sin x - \cos x \right] + C$$

Question 22:

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$\Rightarrow$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \cdot \sec^2\theta\, d\theta = 2\int \theta \cdot \sec^2\theta\, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta \cdot \int \sec^2\theta d\theta - \int \left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right\} d\theta\right]$$

$$= 2\left[\theta \cdot \tan\theta - \int \tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log\left(1+x^2\right)^{-\frac{1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log\left(1+x^2\right)\right] + C$$

$$= 2x\tan^{-1}x - \log\left(1+x^2\right) + C$$

Question 23:

$\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{3}e^{x^2} + C$

Let $I = \int x^2 e^{x^3} dx$

Also, let $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$
$$= \frac{1}{3}\left(e^t\right) + C$$
$$= \frac{1}{3}e^{x^3} + C$$

Hence, the correct answer is A.

Question 24:

$\int e^x \sec x(1+\tan x)dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$
(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

$$\int e^x \sec x(1+\tan x)dx$$

Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x \tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x \tan x = f'(x)$

It is known that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$

$$\therefore I = e^x \sec x + C$$

Hence, the correct answer is B.

Exercise -7.7

Question 1:

$\sqrt{4-x^2}$

Let $I = \int\sqrt{4-x^2}\,dx = \int\sqrt{(2)^2-(x)^2}\,dx$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\therefore I = \frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+C$

$= \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

Question 2:

$\sqrt{1-4x^2}$

Let $I = \int\sqrt{1-4x^2}\,dx = \int\sqrt{(1)^2-(2x)^2}\,dx$

Let $2x = t \Rightarrow 2dx = dt$

$\therefore I = \frac{1}{2}\int\sqrt{(1)^2-(t)^2}\,dt$

It is known that, $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\Rightarrow I = \frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$

$= \frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+C$

$= \frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$

$= \frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$

Question 3:

$\sqrt{x^2+4x+6}$

Let $I = \int\sqrt{x^2+4x+6}\,dx$

$= \int\sqrt{x^2+4x+4+2}\,dx$

$= \int\sqrt{(x^2+4x+4)+2}\,dx$

$= \int\sqrt{(x+2)^2+(\sqrt{2})^2}\,dx$

It is known that, $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$
$$=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

Question 4:

$\sqrt{x^2+4x+1}$

$$\text{Let } I=\int\sqrt{x^2+4x+1}\,dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)-3}\,dx$$
$$=\int\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$$

Question 5:

$\sqrt{1-4x-x^2}$

$$\text{Let } I=\int\sqrt{1-4x-x^2}\,dx$$
$$=\int\sqrt{1-\left(x^2+4x+4-4\right)}\,dx$$
$$=\int\sqrt{1+4-(x+2)^2}\,dx$$
$$=\int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$$

It is known that, $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

Question 6:

$\sqrt{x^2+4x-5}$

$$\begin{aligned}\text{Let } I&=\int\sqrt{x^2+4x-5}\,dx\\&=\int\sqrt{\left(x^2+4x+4\right)-9}\,dx\\&=\int\sqrt{(x+2)^2-(3)^2}\,dx\end{aligned}$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$$

Question 7:

$\sqrt{1+3x-x^2}$

$$\begin{aligned}\text{Let } I&=\int\sqrt{1+3x-x^2}\,dx\\&=\int\sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\,dx\\&=\int\sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\,dx\\&=\int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx\end{aligned}$$

It is known that, $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$$\therefore I = \frac{x - \frac{3}{2}}{2}\sqrt{1 + 3x - x^2} + \frac{13}{4 \times 2}\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$
$$= \frac{2x - 3}{4}\sqrt{1 + 3x - x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x - 3}{\sqrt{13}}\right) + C$$

Question 8:

$\sqrt{x^2 + 3x}$

$$\text{Let } I = \int \sqrt{x^2 + 3x}\,dx$$
$$= \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}}\,dx$$
$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

It is known that, $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

$$\therefore I = \frac{\left(x + \frac{3}{2}\right)}{2}\sqrt{x^2 + 3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$
$$= \frac{(2x + 3)}{4}\sqrt{x^2 + 3x} - \frac{9}{8}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

Question 9:

$\sqrt{1 + \frac{x^2}{9}}$

Let $I = \int \sqrt{1+\frac{x^2}{9}}\,dx = \frac{1}{3}\int \sqrt{9+x^2}\,dx = \frac{1}{3}\int \sqrt{(3)^2+x^2}\,dx$

It is known that, $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

$$\therefore I = \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right] + C$$
$$= \frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log\left|x+\sqrt{x^2+9}\right| + C$$

Question 10:

$\int \sqrt{1+x^2}\,dx$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + C$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{2}{3}} + C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{3}{2}} + C$

D. $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C$

It is known that, $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

$$\therefore \int \sqrt{1+x^2}\,dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + C$$

Hence, the correct answer is A.

Question 11:

$\int \sqrt{x^2-8x+7}\,dx$ is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

$$\begin{aligned}\text{Let } I &= \int\sqrt{x^2-8x+7}\,dx\\ &= \int\sqrt{\left(x^2-8x+16\right)-9}\,dx\\ &= \int\sqrt{(x-4)^2-(3)^2}\,dx\end{aligned}$$

It is known that, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$$

Hence, the correct answer is D.

Exercise -7.8

Question 1:

$$\int_a^b x\,dx$$

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=a$, $b=b$, and $f(x)=x$

$$\therefore \int_a^b x\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+\ldots+a}_{n\text{ times}}\right)+\left(h+2h+3h+\ldots+(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\ldots+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 2:

$$\int_0^5 (x+1)\,dx$$

Let $I=\int_0^5 (x+1)\,dx$

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)\ldots f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=5$, and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)\,dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1\ldots1}_{n\text{ times}}\right)+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

Question 3:

$$\int_2^3 x^2\,dx$$

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)\ldots f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2$, $b=3$, and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\ldots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\ldots\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot2\cdot\frac{1}{n}\right\}+\ldots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(2^2\underset{n\text{ times}}{+\ldots+}2^2\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\ldots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

Question 4:

$$\int_1^4\left(x^2-x\right)dx$$

Let $I = \int_1^4 \left(x^2 - x\right) dx$

$= \int_1^4 x^2\, dx - \int_1^4 x\, dx$

Let $I = I_1 - I_2$, where $I_1 = \int_1^4 x^2 dx$ and $I_2 = \int_1^4 x\, dx$ ...(1)

It is known that,

$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+(n-1)h)\right]$, where $h=\frac{b-a}{n}$

For $I_1 = \int_1^4 x^2 dx$,

$a = 1, b = 4,$ and $f(x) = x^2$

$\therefore h = \frac{4-1}{n} = \frac{3}{n}$

$$I_1 = \int_1^4 x^2 dx = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+...+f(1+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2\cdot\frac{3}{n}\right)^2+...\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2\cdot\frac{3}{n}\right\}+...+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2\cdot(n-1)\cdot 3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(1^2+\underset{n\text{ times}}{...}+1^2\right)+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+...+(n-1)^2\right\}+2\cdot\frac{3}{n}\left\{1+2+...+(n-1)\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$= 3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$= 3[1+3+3]$$

$$= 3[7]$$

$I_1 = 21$ ...(2)

For $I_2 = \int_1^4 x dx$,

$a = 1, b = 4,$ and $f(x) = x$

$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$

$$\therefore I_2=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+...f(a+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+...+(1+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+...+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+...+1}_{n\text{ times}}\right)+\frac{3}{n}\left(1+2+...+(n-1)\right)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=3\left[1+\frac{3}{2}\right]$$

$$=3\left[\frac{5}{2}\right]$$

$$I_2=\frac{15}{2} \qquad ...(3)$$

From equations (2) and (3), we obtain

$$I=I_1+I_2=21-\frac{15}{2}=\frac{27}{2}$$

Question 5:

$$\int_{-1}^{1}e^x dx$$

$$\text{Let } I=\int_{-1}^{1}e^x dx \qquad ...(1)$$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)...f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=-1$, $b=1$, and $f(x)=e^x$

$$\therefore h=\frac{1+1}{n}=\frac{2}{n}$$

$$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+\ldots+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\cdot\frac{2}{n}\right)}+\ldots e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=e^{-1}\times2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{2}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=\frac{e^{-1}\times2\left(e^{2}-1\right)}{\lim_{\frac{2}{n}\to0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times2}$$

$$=e^{-1}\left[\frac{2\left(e^{2}-1\right)}{2}\right] \qquad \left[\lim_{h\to0}\left(\frac{e^{h}-1}{h}\right)=1\right]$$

$$=\frac{e^{2}-1}{e}$$

$$=\left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4\left(x+e^{2x}\right)dx$$

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=4$, and $f(x)=x+e^{2x}$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x+e^{2x}\right)dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f((n-1)h)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}\right)+\left(2h+e^{2\cdot 2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$= 4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8+\frac{4\cdot\left(e^8-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^x-1}{x}=1\right)$$

$$= 8+\frac{e^8-1}{2}$$

$$= \frac{15+e^8}{2}$$

Exercise -7.9

Question 1:

$$\int_{-1}^{1}(x+1)dx$$

$$\text{Let } I = \int_{-1}^{1}(x+1)dx$$

$$\int(x+1)\,dx = \frac{x^2}{2}+x = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(-1)$$
$$= \left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)$$
$$= \frac{1}{2}+1-\frac{1}{2}+1$$
$$= 2$$

Question 2:

$$\int_2^3 \frac{1}{x}dx$$

$$\text{Let } I = \int_2^3 \frac{1}{x}\,dx$$

$$\int \frac{1}{x}\,dx = \log|x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$
$$= \log|3| - \log|2| = \log\frac{3}{2}$$

Question 3:

$$\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right)dx$$

$$\text{Let } I = \int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right)dx$$

$$\int \left(4x^3 - 5x^2 + 6x + 9\right)dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$$
$$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(1)$$

$$I = \left\{2^4 - \frac{5\cdot(2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

Question 4:

$$\int_0^{\frac{\pi}{4}} \sin 2x dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$= -\frac{1}{2}[0 - 1]$$

$$= \frac{1}{2}$$

Question 5:

$\int_0^{\frac{\pi}{2}} \cos 2x \, dx$

Let $I = \int_0^{\frac{\pi}{2}} \cos 2x \, dx$

$\int \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right) = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}\left(\frac{\pi}{2}\right) - \mathrm{F}(0) \\ &= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right] \\ &= \frac{1}{2}\left[\sin\pi - \sin 0\right] \\ &= \frac{1}{2}\left[0 - 0\right] = 0 \end{aligned}$$

Question 6:

$\int_4^5 e^x \, dx$

Let $I = \int_4^5 e^x \, dx$

$\int e^x \, dx = e^x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(5) - \mathrm{F}(4) \\ &= e^5 - e^4 \\ &= e^4(e - 1) \end{aligned}$$

Question 7:

$\int_0^{\frac{\pi}{4}} \tan x \, dx$

Let $I = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = F(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F(0) \\ &= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0| \\ &= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1| \\ &= -\log(2)^{-\frac{1}{2}} \\ &= \frac{1}{2}\log 2 \end{aligned}$$

Question 8:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

$\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| = F(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right) \\ &= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right| \\ &= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right| \\ &= \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right) \end{aligned}$$

Question 9:

$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(1) - \mathrm{F}(0) \\ &= \sin^{-1}(1) - \sin^{-1}(0) \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

Question 10:

$\int_0^1 \frac{dx}{1+x^2}$

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = \mathrm{F}(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(1) - \mathrm{F}(0) \\ &= \tan^{-1}(1) - \tan^{-1}(0) \\ &= \frac{\pi}{4} \end{aligned}$$

Question 11:

$\int_2^3 \frac{dx}{x^2-1}$

Let $I = \int_2^3 \frac{dx}{x^2 - 1}$

$\int \frac{dx}{x^2 - 1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = F(x)$

By second fundamental theorem of calculus, we obtain

$$
\begin{aligned}
I &= F(3) - F(2) \\
&= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right] \\
&= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right] \\
&= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right] \\
&= \frac{1}{2}\left[\log\frac{3}{2}\right]
\end{aligned}
$$

Question 12:

$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$\int \cos^2 x \, dx = \int\left(\frac{1+\cos 2x}{2}\right)dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) = F(x)$

By second fundamental theorem of calculus, we obtain

$$
\begin{aligned}
I &= \left[F\left(\frac{\pi}{2}\right) - F(0)\right] \\
&= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right] \\
&= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right] \\
&= \frac{\pi}{4}
\end{aligned}
$$

Question 13:

$$\int_2^3 \frac{xdx}{x^2+1}$$

$$\text{Let } I = \int_2^3 \frac{x}{x^2+1}dx$$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\log\left(1+x^2\right) = \mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned} I &= \mathrm{F}(3) - \mathrm{F}(2) \\ &= \frac{1}{2}\left[\log\left(1+(3)^2\right) - \log\left(1+(2)^2\right)\right] \\ &= \frac{1}{2}\left[\log(10) - \log(5)\right] \\ &= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2 \end{aligned}$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1}dx$$

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1}dx$$

$$\begin{aligned} \int \frac{2x+3}{5x^2+1}dx &= \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx \\ &= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx \\ &= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx \\ &= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx \\ &= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}} \\ &= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}x\right) \\ &= \mathrm{F}(x) \end{aligned}$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$
$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 xe^{x^2}\,dx$$

Let $I = \int_0^1 xe^{x^2}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1$,

$$\therefore I = \frac{1}{2}\int_0^1 e^t\,dt$$

$$\frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t = F(t)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \frac{1}{2}e - \frac{1}{2}e^0$$
$$= \frac{1}{2}(e-1)$$

Question 16:

$$\int_1^2 \frac{5x^2}{x^2+4x+3}$$

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$

Dividing $5x^2$ by $x^2+4x+3,$ we obtain

$$I=\int_1^2\left\{5-\frac{20x+15}{x^2+4x+3}\right\}dx$$

$$=\int_1^2 5dx-\int_1^2\frac{20x+15}{x^2+4x+3}dx$$

$$=\left[5x\right]_1^2-\int_1^2\frac{20x+15}{x^2+4x+3}dx$$

$$I=5-I_1,\text{ where } I=\int_1^2\frac{20x+15}{x^2+4x+3}dx \qquad ...(1)$$

$$\text{Consider } I_1=\int_1^2\frac{20x+15}{x^2+4x+8}dx$$

$$\text{Let } 20x+15=\text{A}\frac{d}{dx}\left(x^2+4x+3\right)+\text{B}$$

$$=2\text{A}x+\left(4\text{A}+\text{B}\right)$$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

$$\Rightarrow I_1=10\int_1^2\frac{2x+4}{x^2+4x+3}dx-25\int_1^2\frac{dx}{x^2+4x+3}$$

$$\text{Let } x^2+4x+3=t$$

$$\Rightarrow (2x+4)dx=dt$$

$$\Rightarrow I_1=10\int\frac{dt}{t}-25\int\frac{dx}{(x+2)^2-1^2}$$

$$=10\log t-25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$$

$$=\left[10\log\left(x^2+4x+3\right)\right]_1^2-25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$=\left[10\log 15-10\log 8\right]-25\left[\frac{1}{2}\log\frac{3}{5}-\frac{1}{2}\log\frac{2}{4}\right]$$

$$=\left[10\log(5\times 3)-10\log(4\times 2)\right]-\frac{25}{2}\left[\log 3-\log 5-\log 2+\log 4\right]$$

$$=\left[10\log 5+10\log 3-10\log 4-10\log 2\right]-\frac{25}{2}\left[\log 3-\log 5-\log 2+\log 4\right]$$
$$=\left[10+\frac{25}{2}\right]\log 5+\left[-10-\frac{25}{2}\right]\log 4+\left[10-\frac{25}{2}\right]\log 3+\left[-10+\frac{25}{2}\right]\log 2$$
$$=\frac{45}{2}\log 5-\frac{45}{2}\log 4-\frac{5}{2}\log 3+\frac{5}{2}\log 2$$
$$=\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}$$

Substituting the value of I_1 in (1), we obtain

$$I=5-\left[\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}\right]$$
$$=5-\frac{5}{2}\left[9\log\frac{5}{4}-\log\frac{3}{2}\right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}}\left(2\sec^2 x+x^3+2\right)dx$$

$$\text{Let } I=\int_0^{\frac{\pi}{4}}\left(2\sec^2 x+x^3+2\right)dx$$
$$\int\left(2\sec^2 x+x^3+2\right)dx=2\tan x+\frac{x^4}{4}+2x=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\mathrm{F}\left(\frac{\pi}{4}\right)-\mathrm{F}(0)$$
$$=\left\{\left(2\tan\frac{\pi}{4}+\frac{1}{4}\left(\frac{\pi}{4}\right)^4+2\left(\frac{\pi}{4}\right)\right)-\left(2\tan 0+0+0\right)\right\}$$
$$=2\tan\frac{\pi}{4}+\frac{\pi^4}{4^5}+\frac{\pi}{2}$$
$$=2+\frac{\pi}{2}+\frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$$

$$\text{Let } I=\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$$
$$=-\int_0^{\pi}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx$$
$$=-\int_0^{\pi}\cos x\,dx$$
$$\int\cos x\,dx=\sin x=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\mathrm{F}(\pi)-\mathrm{F}(0)$$
$$=\sin\pi-\sin 0$$
$$=0$$

Question 19:

$$\int_0^{2}\frac{6x+3}{x^2+4}dx$$

$$\text{Let } I=\int_0^{2}\frac{6x+3}{x^2+4}dx$$
$$\int\frac{6x+3}{x^2+4}dx=3\int\frac{2x+1}{x^2+4}dx$$
$$=3\int\frac{2x}{x^2+4}dx+3\int\frac{1}{x^2+4}dx$$
$$=3\log\left(x^2+4\right)+\frac{3}{2}\tan^{-1}\frac{x}{2}=\mathrm{F}(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(0)$$
$$= \left\{3\log\left(2^2+4\right)+\frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\}-\left\{3\log(0+4)+\frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$
$$= 3\log 8+\frac{3}{2}\tan^{-1}1-3\log 4-\frac{3}{2}\tan^{-1}0$$
$$= 3\log 8+\frac{3}{2}\left(\frac{\pi}{4}\right)-3\log 4-0$$
$$= 3\log\left(\frac{8}{4}\right)+\frac{3\pi}{8}$$
$$= 3\log 2+\frac{3\pi}{8}$$

Question 20:

$$\int_0^1\left(xe^x+\sin\frac{\pi x}{4}\right)dx$$

Let $I = \int_0^1\left(xe^x+\sin\frac{\pi x}{4}\right)dx$

$$\int\left(xe^x+\sin\frac{\pi x}{4}\right)dx = x\int e^x dx-\int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx+\left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$
$$= xe^x-\int e^x dx-\frac{4\pi}{\pi}\cos\frac{x}{4}$$
$$= xe^x-e^x-\frac{4\pi}{\pi}\cos\frac{x}{4}$$
$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$
$$= \left(1.e^1-e^1-\frac{4}{\pi}\cos\frac{\pi}{4}\right)-\left(0.e^0-e^0-\frac{4}{\pi}\cos 0\right)$$
$$= e-e-\frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right)+1+\frac{4}{\pi}$$
$$= 1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$$

Question 21:

$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$

By second fundamental theorem of calculus, we obtain

$$\begin{aligned}\int_1^{\sqrt{3}} \frac{dx}{1+x^2} &= F(\sqrt{3}) - F(1)\\ &= \tan^{-1}\sqrt{3} - \tan^{-1} 1\\ &= \frac{\pi}{3} - \frac{\pi}{4}\\ &= \frac{\pi}{12}\end{aligned}$$

Hence, the correct answer is D.

Question 22:

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$
$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$$
$$= \frac{1}{6}\tan^{-1}1 - 0$$
$$= \frac{1}{6}\times\frac{\pi}{4}$$
$$= \frac{\pi}{24}$$

Hence, the correct answer is C.

Exercise -7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1} dx$$

$$\int_0^1 \frac{x}{x^2+1} dx$$

Let $x^2+1=t \Rightarrow 2x\,dx = dt$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$
$$= \frac{1}{2}\left[\log|t|\right]_1^2$$
$$= \frac{1}{2}\left[\log 2 - \log 1\right]$$
$$= \frac{1}{2}\log 2$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\, d\phi$$

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^4\phi\cos\phi\, d\phi$

Also, let $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

When $\phi = 0$, $t = 0$ and when $\phi = \frac{\pi}{2}$, $t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\,(1-t^2)^2\, dt$$
$$= \int_0^1 t^{\frac{1}{2}}(1+t^4-2t^2)\,dt$$
$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$$
$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$
$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$
$$= \frac{154+42-132}{231}$$
$$= \frac{64}{231}$$

Question 3:

$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let x = tanθ ⇒ dx = sec^{2}θ dθ

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

When x = 0, θ = 0 and when x = 1, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta \cdot \sec^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta \cdot \sec^2\theta\, d\theta$$

Takingθas first function and sec^{2}θ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int\sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

Question 4:

$\int_0^2 x\sqrt{x+2}\ \left(\text{Put } x+2=t^2\right)$

Let x + 2 = t^2 ⇒ dx = 2tdt

$\int_0^2 x\sqrt{x+2}dx$

When x = 0, $t=\sqrt{2}$ and when x = 2, t = 2

$$\therefore \int_0^2 x\sqrt{x+2}dx = \int_{\sqrt{2}}^2 (t^2-2)\sqrt{t^2}\,2tdt$$
$$= 2\int_{\sqrt{2}}^2 (t^2-2)t^2 dt$$
$$= 2\int_{\sqrt{2}}^2 (t^4-2t^2)dt$$
$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$
$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$
$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$
$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$
$$= \frac{16\left(2+\sqrt{2}\right)}{15}$$
$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Question 5:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Let cos x = t ⇒ −sinx dx = dt

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

When x = 0, t = 1 and when $x=\frac{\pi}{2}, t=0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$
$$= -\left[\tan^{-1} t\right]_1^0$$
$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$
$$= -\left[-\frac{\pi}{4}\right]$$
$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$
$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$
$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$
$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let $\Rightarrow$ dx = dt

$$x-\frac{1}{2}=t$$

When $x=0,\ t=-\frac{1}{2}$ and when $x=2,\ t=\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}=\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$=\left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)}\log\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}-\frac{\log\frac{\sqrt{17}}{2}-\frac{1}{2}}{\log\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3}-\log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$=\frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5}$$

Let x + 1 = t ⇒ dx = dt

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+4} = \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

When x = −1, t = 0 and when x = 1, t = 2

$$\therefore \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2} = \int_{0}^{2} \frac{dt}{t^2+2^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

Question 8:

$$\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

Let 2x = t ⇒ 2dx = dt

$$\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

When x = 1, t = 2 and when x = 2, t = 4

$$\therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{1}{2}\int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt = \int_2^4 e^t \left[f(t) + f'(t)\right] dt$$

$$= \left[e^t f(t)\right]_2^4$$

$$= \left[e^t \cdot \frac{2}{t}\right]_2^4$$

$$= \left[\frac{e^t}{t}\right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2\left(e^2 - 2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$ is

A. 6

B. 0

C. 3

D. 4

Let $I = \int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$

Also, let $x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

When $x = \frac{1}{3}$, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta - \sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1 - \sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^4\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^2\theta \sin^2\theta} \cos\theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}} \operatorname{cosec}^2\theta \, d\theta$$

Let cotθ = t ⇒ −cosec2θ dθ= dt

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\cot\theta\right)^{\frac{5}{3}} \operatorname{cosec}^2\theta \, d\theta$$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$

$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$$= 3 \times 2$$

$$= 6$$

Hence, the correct answer is A.

Question 10:

If $f(x)=\int_0^x t\sin t\,dt$, then $f'(x)$ is

A. cos x + x sin x

B. x sin x

C. x cos x

D. sin x + x cos x

$$f(x)=\int_0^x t\sin t\,dt$$

Integrating by parts, we obtain

$$\begin{aligned} f(x) &= t\int_0^x \sin t\,dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt \\ &= \left[t(-\cos t)\right]_0^x-\int_0^x(-\cos t)\,dt \\ &= \left[-t\cos t+\sin t\right]_0^x \\ &= -x\cos x+\sin x \end{aligned}$$

$$\begin{aligned} \Rightarrow f'(x) &= -\left[\{x(-\sin x)\}+\cos x\right]+\cos x \\ &= x\sin x-\cos x+\cos x \\ &= x\sin x \end{aligned}$$

Hence, the correct answer is B.

Exercise :- 7.11

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

Answer :

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Answer :

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}} dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \qquad \ldots(2)$$

Adding 1 and 2 we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 4

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \qquad \ldots(2)$$

Adding 1 and 2 we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 5

$$\int_{-5}^{5} |x+2| \, dx$$

Let $I = \int_{-5}^{5} |x+2| dx$

It can be seen that (x + 2) ≤ 0 on [-5,-2] and (x + 2) ≥ 0 on [-2,5]

$$\therefore I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$= 29$$

Question 6

$\int_{2}^{8} |x-5| dx$

Let $I = \int_{2}^{8} |x-5| dx$

It can be seen that (x - 5) ≤ 0 on [2,5] and (x - 5) ≥ 0 on [5,8]

$$I = \int_{2}^{5} -(x-5)dx + \int_{2}^{8} (x-5)dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$= 9$$

Question 7

$\int_{0}^{1} x(1-x)^n dx$

$$\text{Let } I=\int_0^1 x(1-x)^n\,dx$$
$$\therefore I=\int_0^1 (1-x)(1-(1-x))^n\,dx$$
$$=\int_0^1 (1-x)(x)^n\,dx$$
$$=\int_0^1 \left(x^n-x^{n+1}\right)dx$$
$$=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1 \qquad \left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$
$$=\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$$
$$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}$$
$$=\frac{1}{(n+1)(n+2)}$$

Question 8

$$\int_0^{\frac{\pi}{4}} \log\,(1+\tan x)dx$$

$$\text{Let } I=\int_0^{\frac{\pi}{4}} \log\,(1+\tan x)\,dx \qquad \ldots(1)$$
$$\therefore I=\int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \qquad \left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$
$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$$
$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan}\right\}dx$$
$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\,\frac{2}{(1+\tan x)}dx$$
$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log 2\,dx-\int_0^{\frac{\pi}{4}} \log\,(1+\tan x)\,dx$$
$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log 2dx-I \qquad [\text{From } (1)]$$
$$\Rightarrow 2I=[x\log 2]_0^{\frac{\pi}{4}}$$
$$\Rightarrow 2I=\frac{\pi}{4}\log 2$$
$$\Rightarrow I=\frac{\pi}{8}\log 2$$

Question 9

$$\int_0^2 x\sqrt{2-x}\,dx$$

Let $I = \int_0^2 x\sqrt{2-x}\,dx$

$I = \int_0^2 (2-x)\sqrt{x}\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$= \int_0^2 \left\{2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right\}dx$

$= \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$

$= \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$

$= \frac{4\times 2\sqrt{2}}{3} - \frac{2}{5}\times 4\sqrt{2}$

$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$

$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$

$= \frac{16\sqrt{2}}{15}$

Question 10

$\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$

Let $I = \int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log\sin x - \log(2\sin x\cos x)\}\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log\sin x - \log\sin x - \log\cos x - \log 2\}\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log\sin x - \log\cos x - \log 2\}\,dx \qquad \text{...(1)}$

It is known that, $\left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log\cos x - \log\sin x - \log 2\}\,dx \qquad \text{...(2)}$

Adding 1 and 2 we obtain

$$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)dx$$

$$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1\,dx$$

$$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$$

$$\Rightarrow I=\frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$$

$$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$$

Question 11:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$$

Answer :

$$\text{Let } I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$$

As $\sin^2(-x)=(\sin(-x))^2=(-\sin x)^2=\sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a}f(x)\,dx=2\int_0^{a}f(x)\,dx$

$$I=2\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$$

$$=2\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}dx$$

$$=\int_0^{\frac{\pi}{2}}(1-\cos 2x)\,dx$$

$$=\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$

Question 12:

$$\int_0^{\pi} \frac{x\,dx}{1+\sin x}$$

Answer :

$$\text{Let } I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

Continued Further

$$2I = \int_0^\pi \frac{\pi}{1+\sin x} dx$$
$$\Rightarrow 2I = \pi \int_0^\pi \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$
$$\Rightarrow 2I = \pi \int_0^\pi \frac{1-\sin x}{\cos^2 x} dx$$
$$\Rightarrow 2I = \pi \int_0^\pi \{\sec^2 x - \tan x \sec x\} dx$$
$$\Rightarrow 2I = \pi [\tan x - \sec x]_0^\pi$$
$$\Rightarrow 2I = \pi [2]$$
$$\Rightarrow I = \pi$$

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Question 13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$$

Answer :

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx \qquad \ldots(1)$$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^2 x$ is an odd function. It is known that, if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx = 0$$

Question 14:

$$\int_0^{2\pi} \cos^5 x dx$$

Answer :

$$\text{Let } I = \int_0^{2\pi} \cos^5 x dx \qquad \ldots(1)$$
$$\cos^5(2\pi - x) = \cos^5 x$$

It is known that,

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$
$$= 0 \text{ if } f(2a-x) = -f(x)$$

$$\therefore I = 2\int_0^{\pi} \cos^5 x\,dx$$

$$\Rightarrow I = 2(0) = 0 \qquad \left[\cos^5(\pi - x) = -\cos^5 x\right]$$

Question 15:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx$$

Answer :

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}\,dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left(\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x\cos x}\,dx$$
$$\Rightarrow I = 0$$

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Question 16:

$$\int_0^{\pi} \log(1+\cos x)\,dx$$

Answer :

$$\text{Let } I = \int_0^{\pi} \log(1+\cos x)\,dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\pi} \log\left(1+\cos\left(\pi - x\right)\right) dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \log\left(1-\cos x\right) dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \left\{\log\left(1+\cos x\right) + \log\left(1-\cos x\right)\right\} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log\left(1-\cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x \, dx$$

$$\Rightarrow 2I = 2\int_0^{\pi} \log \sin x \, dx$$

$$\Rightarrow I = \int_0^{\pi} \log \sin x \, dx \qquad \ldots(3)$$

sin (p - x) = sin x

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad \ldots(4)$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2\int_0^{\frac{\pi}{2}} \log \cos x \, dx \qquad \ldots(5)$$

Adding (4) and (5), we obtain

$$2I = 2\int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x + \log 2 - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\log 2 \sin x \cos x - \log 2\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Let 2x = t ? 2dx = dt

When x = 0, t = 0
and when x = $\frac{p}{2}$, t = p
? I = $\frac{1}{2}$?$_0^p$ log sin tdt- $\frac{p}{2}$ log 2
? I = $\frac{I}{2}$ -$\frac{p}{2}$ log 2 [from 3]
? $\frac{I}{2}$ = -$\frac{p}{2}$ log 2
? I = -$p$log 2

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Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Answer :

$$\text{Let } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(1)$$

It is known that, $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$
$$\Rightarrow 2I = \int_0^a 1 \cdot dx$$
$$\Rightarrow 2I = [x]_0^a$$
$$\Rightarrow 2I = a$$
$$\Rightarrow I = \frac{a}{2}$$

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Question 18:

$$\int_0^4 |x-1|\, dx$$

Answer :

$$I = \int_0^4 |x-1|\, dx$$

It can be seen that, (x - 1) = 0 when 0 = x = 1 and (x - 1) = 0 when 1 = x = 4

$$
\begin{aligned}
I &= \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)\\
&= \int_0^1 -(x-1)\,dx + \int_0^4 (x-1)\,dx\\
&= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4\\
&= 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1\\
&= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1\\
&= 5
\end{aligned}
$$

Question 19:

Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Answer :

$$\text{Let } I = \int_0^a f(x)g(x)\,dx \qquad \ldots(1)$$

$$
\begin{aligned}
&\Rightarrow I = \int_0^a f(a-x)g(a-x)\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)\\
&\Rightarrow I = \int_0^a f(x)g(a-x)\,dx \qquad \ldots(2)
\end{aligned}
$$

Adding (1) and (2), we obtain

$$
\begin{aligned}
&2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}\,dx\\
&\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}\,dx\\
&\Rightarrow 2I = \int_0^a f(x)\times 4\,dx \qquad [g(x) + g(a-x) = 4]\\
&\Rightarrow I = 2\int_0^a f(x)\,dx
\end{aligned}
$$

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Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A. 0

B. 2

C. p

D. 1

Answer :

$$\text{Let } I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx$$

$$\Rightarrow I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^3dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tan^5 xdx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\cdot dx$$

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx$ and

if $f(x)$ is an odd function, then $\int_{-a}^{a}f(x)dx=0$

$$I=0+0+0+2\int_0^{\frac{\pi}{2}}1\cdot dx$$

$$=2[x]_0^{\frac{\pi}{2}}$$

$$=\frac{2\pi}{2}$$

$$=\pi$$

Hence, the correct answer is C.

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Question 21:

The value of $\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. -2

Answer :

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right]dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right\}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1\,dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0\,dx$$

$$\Rightarrow I = 0$$

Hence, the correct answer is C.