Exercise- 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y.$ Then $\sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sin^{-1}$ is

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}.$

Therefore, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}.$

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y.$ Then, $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0, \pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$

Let $\operatorname{cosec}^{-1}(2) = y$. Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4:

Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$. Then, $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $-\frac{\pi}{3}$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Let $\tan^{-1}(-1) = y$. Then, $\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1$.

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\sec^{-1}$ is

$[0,\pi] - \left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$.

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Question 8:

Find the principal value of $\cot^{-1}\left(\sqrt{3}\right)$

Let $\cot^{-1}\left(\sqrt{3}\right) = y$. Then, $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\cot^{-1}$ is $(0,\pi)$ and

$\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$.

Therefore, the principal value of $\cot^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{6}$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$. Then, $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$ and

$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Let $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = y$. Then, $\operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\left(\frac{\pi}{4}\right) = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$ and $\operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

Question 11:

Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Let $\tan^{-1}(1) = x$. Then, $\tan x = 1 = \tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1) = \frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = z$. Then, $\sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$

$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$. Then, $\sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Question 13:

Find the value of if $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ **(B)** $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ **(D)** $-\frac{\pi}{2} < y < \frac{\pi}{2}$

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Question 14:

Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π **(B)** $-\frac{\pi}{3}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{2\pi}{3}$

Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = -\sec\left(\frac{\pi}{3}\right) = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0,\pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

Hence, $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

Exercise- 2.2

Question 1:

Prove $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right),\ x\in\left[-\frac{1}{2},\ \frac{1}{2}\right]$

To prove: $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right),\ x\in\left[-\frac{1}{2},\ \frac{1}{2}\right]$

Let $x = \sin\theta$. Then, $\sin^{-1}x=\theta$.

We have,

R.H.S. = $\sin^{-1}\left(3x-4x^3\right)=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)$

$=\sin^{-1}\left(\sin 3\theta\right)$

$= 3\theta$

$=3\sin^{-1}x$

= L.H.S.

Question 2:

Prove $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right),\ x\in\left[\frac{1}{2},\ 1\right]$

To prove: $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right),\ x\in\left[\frac{1}{2},\ 1\right]$

Let $x = \cos\theta$. Then, $\cos^{-1}x=\theta$.

We have,

$$\begin{aligned}\text{R.H.S.}&=\cos^{-1}\left(4x^3-3x\right)\\&=\cos^{-1}\left(4\cos^3\theta-3\cos\theta\right)\\&=\cos^{-1}\left(\cos 3\theta\right)\\&=3\theta\\&=3\cos^{-1}x\\&=\text{L.H.S.}\end{aligned}$$

Question 3:

Prove $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

To prove: $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

$$\text{L.H.S.}=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$$

$$=\tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\frac{\frac{48+77}{11\times 24}}{\frac{11\times 24-14}{11\times 24}}$$

$$=\tan^{-1}\frac{48+77}{264-14}=\tan^{-1}\frac{125}{250}=\tan^{-1}\frac{1}{2}=\text{R.H.S.}$$

Question 4:

Prove $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

To prove: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$= \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0$$

$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

Question 6:

Write the function in the simplest form:

$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$

$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$

Put $x=\text{cosec}\,\theta \Rightarrow \theta=\text{cosec}^{-1}x$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\text{cosec}^2\theta-1}}$$

$$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)=\tan^{-1}(\tan\theta)$$

$$=\theta=\text{cosec}^{-1}x=\frac{\pi}{2}-\sec^{-1}x \qquad \left[\text{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}\right]$$

Question 7:

Write the function in the simplest form:

$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ 0<x<\pi$$

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}},\ |x|<a$$

$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$

Put $x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$

$= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$

$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}$

Question 10:

Write the function in the simplest form:

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right),\ a>0;\ \frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

Put $x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^2\cdot a\tan\theta - a^3\tan^3\theta}{a^3-3a\cdot a^2\tan^2\theta}\right)$

$= \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$

$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta$

$= 3\tan^{-1}\frac{x}{a}$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Let $\sin^{-1}\frac{1}{2}=x$. Then, $\sin x=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]=\tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$=\tan^{-1}1=\frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$

$$\cot\left(\tan^{-1}a+\cot^{-1}a\right)$$

$$=\cot\left(\frac{\pi}{2}\right) \qquad \left[\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$$

$$=0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right],\ |x|<1,\ y>0 \text{ and } xy<1$

Let $x = \tan\theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}\left(\sin 2\theta\right)=2\theta=2\tan^{-1}x$$

Let $y = \tan\Phi$. Then, $\Phi = \tan^{-1} y$.

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)=\cos^{-1}(\cos 2\phi)=2\phi=2\tan^{-1}y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$=\tan\frac{1}{2}\left[2\tan^{-1}x+2\tan^{-1}y\right]$$

$$=\tan\left[\tan^{-1}x+\tan^{-1}y\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left[\sin(A+B)=\sin A\cos B+\cos A\sin B\right]$$

$$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \quad \ldots(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$.

Then, $\sin y=\frac{1}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$.

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \quad \ldots(2)$$

Let $\cos^{-1}x=z$.

Then, $\cos z=x\Rightarrow \sin z=\sqrt{1-x^2}\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right)$.

$$\therefore \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right) \quad \ldots(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\cdot\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\cdot\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1-x^2}=5-x$$

On squaring both sides, we get:

$$(4)(6)(1-x^2)=25+x^2-10x$$
$$\Rightarrow 24-24x^2=25+x^2-10x$$
$$\Rightarrow 25x^2-10x+1=0$$
$$\Rightarrow (5x-1)^2=0$$
$$\Rightarrow (5x-1)=0$$
$$\Rightarrow x=\frac{1}{5}$$

Hence, the value of x is $\frac{1}{5}$.

Question 15:

If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x.

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, the value of x is $\pm\frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \text{ where } -\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

Let $\sin^{-1}\frac{3}{5} = x$. Then, $\sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \frac{4}{5} \Rightarrow \sec x = \frac{5}{4}$.

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \quad \text{...(i)}$$

$$\text{Now, } \cot^{-1}\frac{3}{2} = \tan^{-1}\frac{2}{3} \quad \text{...(ii)} \qquad \left[\tan^{-1}\frac{1}{x} = \cot^{-1} x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) \qquad [\text{Using (i) and (ii)}]$$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) \qquad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad \left[\cos(2\pi + x) = \cos x\right]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \text{ where } \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** 1

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \sin^{-1}\left(\frac{-1}{2}\right)=\frac{-\pi}{6}$

$$\therefore \sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\left(\frac{3\pi}{6}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

The correct answer is D.

Question 21:

Find the values of $\tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)$ is equal to

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$

Let $\tan^{-1}\sqrt{3}=x$. Then, $\tan x=\sqrt{3}=\tan\frac{\pi}{3}$ where $\frac{\pi}{3}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3}=\frac{\pi}{3}$

Let $\cot^{-1}\left(-\sqrt{3}\right)=y$.

$$\text{Then, }\cot y=-\sqrt{3}=-\cot\left(\frac{\pi}{6}\right)=\cot\left(\pi-\frac{\pi}{6}\right)=\cot\frac{5\pi}{6}\text{ where }\frac{5\pi}{6}\in(0,\ \pi).$$

The range of the principal value branch of $\cot^{-1}$ is $(0,\ \pi)$.

$\therefore \cot^{-1}\left(-\sqrt{3}\right)=\frac{5\pi}{6}$

$$\therefore \tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)=\frac{\pi}{3}-\frac{5\pi}{6}=\frac{2\pi-5\pi}{6}=\frac{-3\pi}{6}=-\frac{\pi}{2}$$

The correct answer is B.

Miscellaneous

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in [0, \pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left[\tan\left(2\pi-\frac{5\pi}{6}\right)\right] \qquad \left[\tan(2\pi-x)=-\tan x\right]$$

$$=\tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right]=\tan^{-1}\left[\tan\left(-\frac{5\pi}{6}\right)\right]=\tan^{-1}\left[\tan\left(\pi-\frac{5\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right],\text{ where }\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$$

Question 3:

Prove $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}$.

$$\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$$

$$\therefore \tan x=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}\Rightarrow \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$$

Now, we have:

$$\text{L.H.S.}=2\sin^{-1}\frac{3}{5}=2\tan^{-1}\frac{3}{4}$$

$$=\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \qquad \left[2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$=\tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right)=\tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$=\tan^{-1}\frac{24}{7}=\text{R.H.S.}$$

Question 4:

Prove $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Let $\sin^{-1}\frac{8}{17}=x$. Then, $\sin x=\frac{8}{17}\Rightarrow \cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{\frac{225}{289}}=\frac{15}{17}$.

$\therefore \tan x=\frac{8}{15}\Rightarrow x=\tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15}$ ...(1)

Now, let $\sin^{-1}\frac{3}{5}=y$. Then, $\sin y=\frac{3}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan y=\frac{3}{4}\Rightarrow y=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(2)

Now, we have:

$$\begin{aligned}
\text{L.H.S.} &= \sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}\\
&= \tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4} && [\text{Using (1) and (2)}]\\
&= \tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\\
&= \tan^{-1}\left(\frac{32+45}{60-24}\right) && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\
&= \tan^{-1}\frac{77}{36}=\text{R.H.S.}
\end{aligned}$$

Question 5:

Prove $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$

Let $\cos^{-1}\frac{4}{5} = x$. Then, $\cos x = \frac{4}{5} \Rightarrow \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$.

$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$

$\therefore \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4} \qquad ...(1)$

Now, let $\cos^{-1}\frac{12}{13} = y$. Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}$.

$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12} \qquad ...(2)$

Let $\cos^{-1}\frac{33}{65} = z$. Then, $\cos z = \frac{33}{65} \Rightarrow \sin z = \frac{56}{65}$.

$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1}\frac{56}{33}$

$\therefore \cos^{-1}\frac{33}{65} = \tan^{-1}\frac{56}{33} \qquad ...(3)$

Now, we will prove that:

$$\begin{aligned}
\text{L.H.S.} &= \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} \\
&= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12} && \left[\text{Using (1) and (2)}\right] \\
&= \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}.\frac{5}{12}} && \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right] \\
&= \tan^{-1}\frac{36+20}{48-15} \\
&= \tan^{-1}\frac{56}{33} \\
&= \tan^{-1}\frac{56}{33} && \left[\text{by (3)}\right] \\
&= \text{R.H.S.}
\end{aligned}$$

Question 6:

Prove $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(1)

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$.

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12}$ ...(2)

Let $\sin^{-1}\frac{56}{65}=z$. Then, $\sin z=\frac{56}{65}\Rightarrow \cos z=\frac{33}{65}$.

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \sin^{-1}\frac{56}{65}=\tan^{-1}\frac{56}{33}$ ...(3)

Now, we have:

$$\begin{aligned}\text{L.H.S.}&=\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}\\&=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{3}{4} && \left[\text{Using (1) and (2)}\right]\\&=\tan^{-1}\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}.\frac{3}{4}} && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\&=\tan^{-1}\frac{20+36}{48-15}\\&=\tan^{-1}\frac{56}{33}\\&=\sin^{-1}\frac{56}{65}=\text{R.H.S.} && \left[\text{Using (3)}\right]\end{aligned}$$

Question 7:

Prove $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Let $\sin^{-1}\frac{5}{13}=x$. Then, $\sin x=\frac{5}{13}\Rightarrow\cos x=\frac{12}{13}$.

$\therefore\tan x=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$

$\therefore\sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}\qquad ...(1)$

Let $\cos^{-1}\frac{3}{5}=y$. Then, $\cos y=\frac{3}{5}\Rightarrow\sin y=\frac{4}{5}$.

$\therefore\tan y=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$

$\therefore\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3}\qquad ...(2)$

Using (1) and (2), we have

$$\begin{aligned}\text{R.H.S.}&=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\\&=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\\&=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)\qquad\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\&=\tan^{-1}\left(\frac{15+48}{36-20}\right)\\&=\tan^{-1}\frac{63}{16}\\&=\text{L.H.S.}\end{aligned}$$

Question 8:

Prove $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

$$\text{L.H.S.} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1} 1$$

$$= \frac{\pi}{4} = \text{R.H.S.}$$

Question 9:

Prove $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in [0,\ 1]$

Let $x = \tan^2\theta$. Then, $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}\times 2\theta = \theta = \tan^{-1}\sqrt{x} = \text{L.H.S.}$$

Question 10:

Prove $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2},\ x \in \left(0,\ \frac{\pi}{4}\right)$

Consider $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \qquad \text{(by rationalizing)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$

$$=\frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

$$\therefore \text{L.H.S.}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=\text{R.H.S.}$$

Question 11:

Prove $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x,\ -\dfrac{1}{\sqrt{2}}\le x\le 1$ [**Hint:** put $x = \cos 2\theta$]

Put $x=\cos 2\theta$ so that $\theta=\frac{1}{2}\cos^{-1}x$. Then, we have:

$$\text{L.H.S.}=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$=\tan^{-1}1-\tan^{-1}(\tan\theta) \qquad \left[\tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-\theta=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x=\text{R.H.S.}$$

Question 12:

Prove $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

$$\text{L.H.S.}=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$$
$$=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$
$$=\frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad . \qquad ....(1) \quad \left[\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3}=x$. Then, $\cos x=\frac{1}{3}\Rightarrow \sin x=\sqrt{1-\left(\frac{1}{3}\right)^2}=\frac{2\sqrt{2}}{3}$.

$$\therefore x=\sin^{-1}\frac{2\sqrt{2}}{3}\Rightarrow \cos^{-1}\frac{1}{3}=\sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}=\text{R.H.S.}$$

Question 13:

Solve $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$$
$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2\operatorname{cosec} x) \qquad \left[2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$
$$\Rightarrow \frac{2\cos x}{1-\cos^2 x}=2\operatorname{cosec} x$$
$$\Rightarrow \frac{2\cos x}{\sin^2 x}=\frac{2}{\sin x}$$
$$\Rightarrow \cos x=\sin x$$
$$\Rightarrow \tan x=1$$

$$\therefore x=\frac{\pi}{4}$$

Question 14:

Solve $\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x,(x>0)$

$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x$ $\left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$

$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$

$\Rightarrow x=\tan\frac{\pi}{6}$

$\therefore x=\frac{1}{\sqrt{3}}$

Question 15:

Solve $\sin\left(\tan^{-1}x\right),\ |x|<1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ **(B)** $\frac{1}{\sqrt{1-x^2}}$ **(C)** $\frac{1}{\sqrt{1+x^2}}$ **(D)** $\frac{x}{\sqrt{1+x^2}}$

Let $\tan^{-1}x=y$. Then, $\tan y=x\Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$.

$\therefore y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)=\frac{x}{\sqrt{1+x^2}}$

The correct answer is D.

Question 16:

Solve $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, then x is equal to

(A) $0,\ \frac{1}{2}$ **(B)** $1,\ \frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$

$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow -2\sin^{-1}x=\frac{\pi}{2}-\sin^{-1}(1-x)$

$\Rightarrow -2\sin^{-1}x=\cos^{-1}(1-x) \qquad ...(1)$

Let $\sin^{-1}x=\theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta=\sqrt{1-x^2}$.

$\therefore \theta=\cos^{-1}\left(\sqrt{1-x^2}\right)$

$\therefore \sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)$

Therefore, from equation (1), we have

$$-2\cos^{-1}\left(\sqrt{1-x^2}\right)=\cos^{-1}(1-x)$$

Put $x = \sin y$. Then, we have:

$-2\cos^{-1}\left(\sqrt{1-\sin^2 y}\right)=\cos^{-1}(1-\sin y)$

$\Rightarrow -2\cos^{-1}(\cos y)=\cos^{-1}(1-\sin y)$

$\Rightarrow -2y=\cos^{-1}(1-\sin y)$

$\Rightarrow 1-\sin y=\cos(-2y)=\cos 2y$

$\Rightarrow 1-\sin y=1-2\sin^2 y$

$\Rightarrow 2\sin^2 y-\sin y=0$

$\Rightarrow \sin y(2\sin y-1)=0$

$\Rightarrow \sin y=0 \text{ or } \frac{1}{2}$

$\therefore x=0 \text{ or } x=\frac{1}{2}$

But, when $x=\frac{1}{2}$, it can be observed that:

$$\begin{aligned}\text{L.H.S.}&=\sin^{-1}\left(1-\frac{1}{2}\right)-2\sin^{-1}\frac{1}{2}\\&=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\frac{1}{2}\\&=-\sin^{-1}\frac{1}{2}\\&=-\frac{\pi}{6}\neq\frac{\pi}{2}\neq\text{R.H.S.}\end{aligned}$$

$\therefore x = \frac{1}{2}$ is not the solution of the given equation.

Thus, $x = 0$.

Hence, the correct answer is **C**

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ **(B).** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{-3\pi}{4}$

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$$

$$= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] \qquad \left[\tan^{-1} y - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, the correct answer is **C**.