## Exercise-11.1

**Question 1:**

If a line makes angles 90°, 135°, 45° with *x*, *y* and *z*-axes respectively, find its direction cosines.

Let direction cosines of the line be *l*, *m*, and *n*.

$$l = \cos 90° = 0$$
$$m = \cos 135° = -\frac{1}{\sqrt{2}}$$
$$n = \cos 45° = \frac{1}{\sqrt{2}}$$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$, and $\frac{1}{\sqrt{2}}$.

**Question 2:**

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Let the direction cosines of the line make an angle $\alpha$ with each of the coordinate axes.

$\therefore l = \cos\alpha, m = \cos\alpha, n = \cos\alpha$

$$l^2 + m^2 + n^2 = 1$$
$$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$
$$\Rightarrow 3\cos^2\alpha = 1$$
$$\Rightarrow \cos^2\alpha = \frac{1}{3}$$
$$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes, are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$, and $\pm\frac{1}{\sqrt{3}}$.

**Question 3:**

If a line has the direction ratios −18, 12, −4, then what are its direction cosines?

If a line has direction ratios of −18, 12, and −4, then its direction cosines are

$$\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$$

i.e., $\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$

$\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

Thus, the direction cosines are $-\frac{9}{11}, \frac{6}{11}$, and $\frac{-2}{11}$.

Question 4:

Show that the points (2, 3, 4), (−1, −2, 1), (5, 8, 7) are collinear.

The given points are A (2, 3, 4), B (− 1, − 2, 1), and C (5, 8, 7).

It is known that the direction ratios of line joining the points, $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, are given by, $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$.

The direction ratios of AB are (−1 − 2), (−2 − 3), and (1 − 4) i.e., −3, −5, and −3.

The direction ratios of BC are (5 − (− 1)), (8 − (− 2)), and (7 − 1) i.e., 6, 10, and 6.

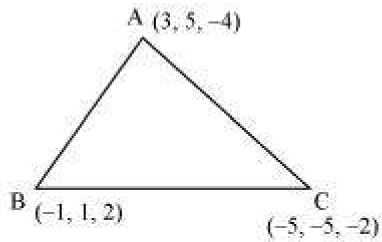
It can be seen that the direction ratios of BC are −2 times that of AB i.e., they are proportional.

Therefore, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

Question 5:

Find the direction cosines of the sides of the triangle whose vertices are (3, 5, − 4), (− 1, 1, 2) and (− 5, − 5, − 2)

The vertices of ΔABC are A (3, 5, −4), B (−1, 1, 2), and C (−5, −5, −2).




The direction ratios of side AB are (−1 − 3), (1 − 5), and (2 − (−4)) i.e., −4, −4, and 6.

$$\text{Then,} \sqrt{(-4)^2+(-4)^2+(6)^2} = \sqrt{16+16+36}$$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

Therefore, the direction cosines of AB are

$$\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$
$$\frac{-4}{2\sqrt{17}}, -\frac{4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}$$
$$\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

The direction ratios of BC are (−5 − (−1)), (−5 − 1), and (−2 − 2) i.e., −4, −6, and −4.

Therefore, the direction cosines of BC are

$$\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$$
$$\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$$

The direction ratios of CA are (−5 − 3), (−5 − 5), and (−2 − (−4)) i.e., −8, −10, and 2.

Therefore, the direction cosines of AC are

$$\frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{-5}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}}$$

$$\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}$$

## Exercise-11.2

**Question 1:**

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Two lines with direction cosines, $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$, are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

**(i)** For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{12}{13}\times\frac{4}{13} + \left(\frac{-3}{13}\right)\times\frac{12}{13} + \left(\frac{-4}{13}\right)\times\frac{3}{13} \\ &= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

**(ii)** For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13}\times\frac{3}{13} + \frac{12}{13}\times\left(\frac{-4}{13}\right) + \frac{3}{13}\times\frac{12}{13} \\ &= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

**(iii)** For the lines with direction cosines, $\frac{3}{13},\frac{-4}{13},\frac{12}{13}$ and $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$, we obtain

$$\begin{aligned}l_1l_2+m_1m_2+n_1n_2&=\left(\frac{3}{13}\right)\times\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\times\left(\frac{-3}{13}\right)+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)\\&=\frac{36}{169}+\frac{12}{169}-\frac{48}{169}\\&=0\end{aligned}$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points (1, −1, 2) (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Let AB be the line joining the points, (1, −1, 2) and (3, 4, − 2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6).

The direction ratios, $a_1$, $b_1$, $c_1$, of AB are (3 − 1), (4 − (−1)), and (−2 − 2) i.e., 2, 5, and −4.

The direction ratios, $a_2$, $b_2$, $c_2$, of CD are (3 − 0), (5 − 3), and (6 −2) i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (−1, −2, 1) and (1, 2, 5).

The directions ratios, $a_1$, $b_1$, $c_1$, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and −4.

The direction ratios, $a_2$, $b_2$, $c_2$, of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

**Question 4:**

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda$ is a constant.

$$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

This is the required equation of the line.

**Question 5:**

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction $\hat{i}+2\hat{j}-\hat{k}$.

It is given that the line passes through the point with position vector

$\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ ...(1)

$\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ ...(2)

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}$

$\Rightarrow \vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$

This is the required equation of the line in vector form.

$\vec{r}=x\hat{i}-y\hat{j}+z\hat{k}$

$\Rightarrow x\hat{i}-y\hat{j}+z\hat{k}=(\lambda+2)\hat{i}+(2\lambda-1)\hat{j}+(-\lambda+4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point

(−2, 4, −5) and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

It is given that the line passes through the point (−2, 4, −5) and is parallel to $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

The direction ratios of the line, $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$, are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Therefore, its direction ratios are $3k$, $5k$, and $6k$, where $k \neq 0$

It is known that the equation of the line through the point $(x_1, y_1, z_1)$ and with direction ratios, $a, b, c$, is given by $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$
$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

**Question 7:**

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

The Cartesian equation of the line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \qquad \dots(1)$$

The given line passes through the point (5, −4, 6). The position vector of this point is $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

This is the required equation of the given line in vector form.

**Question 8:**

Find the vector and the Cartesian equations of the lines that pass through the origin and (5, −2, 3).

The required line passes through the origin. Therefore, its position vector is given by,

$\vec{a} = \vec{0}$ ... (1)

The direction ratios of the line through origin and (5, −2, 3) are

(5 − 0) = 5, (−2 − 0) = −2, (3 − 0) = 3

The line is parallel to the vector given by the equation, $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is, $\vec{r} = \vec{a} + \lambda\vec{b},\ \lambda \in R$

$$\Rightarrow \vec{r} = \vec{0} + \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$
$$\Rightarrow \vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

The equation of the line through the point ($x_1$, $y_1$, $z_1$) and direction ratios $a$, $b$, $c$ is given by, $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x - 0}{5} = \frac{y - 0}{-2} = \frac{z - 0}{3}$$
$$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

**Question 9:**

Find the vector and the Cartesian equations of the line that passes through the points (3, −2, −5), (3, −2, 6).

Let the line passing through the points, P (3, −2, −5) and Q (3, −2, 6), be PQ.

Since PQ passes through P (3, −2, −5), its position vector is given by,

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

The direction ratios of PQ are given by,

(3 − 3) = 0, (−2 + 2) = 0, (6 + 5) = 11

The equation of the vector in the direction of PQ is

$\vec{b} = 0.\hat{i} - 0.\hat{j} + 11\hat{k} = 11\hat{k}$

The equation of PQ in vector form is given by, $\vec{r} = \vec{a} + \lambda\vec{b},\ \lambda \in R$

$\Rightarrow \vec{r} = \left(3\hat{i} - 2\hat{j} - 5\hat{k}\right) + 11\lambda\hat{k}$

The equation of PQ in Cartesian form is

$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ i.e.,

$\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$

Question 10:

Find the angle between the following pairs of lines:

**(i)** $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

**(ii)** $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - 2\hat{k}\right)$ and

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

**(i)** Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, $\cos Q = \left|\dfrac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

The given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

$$\therefore |\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|\vec{b}_2| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(3\hat{i} + 2\hat{j} + 6\hat{k}\right) \cdot \left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \\ &= 3\times1 + 2\times2 + 6\times2 \\ &= 3 + 4 + 12 \\ &= 19\end{aligned}$$

$$\Rightarrow \cos Q = \frac{19}{7\times3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$$

**(ii)** The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$$\therefore |\vec{b}_1| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b}_2| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(\hat{i} - \hat{j} - 2\hat{k}\right) \cdot \left(3\hat{i} - 5\hat{j} - 4\hat{k}\right) \\ &= 1\cdot3 - 1(-5) - 2(-4) \\ &= 3 + 5 + 8 \\ &= 16\end{aligned}$$

$$\cos Q=\left|\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

$$\Rightarrow \cos Q=\frac{16}{\sqrt{6}\cdot 5\sqrt{2}}=\frac{16}{\sqrt{2}\cdot\sqrt{3}\cdot 5\sqrt{2}}=\frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q=\frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

**Question 11:**

Find the angle between the following pairs of lines:

**(i)** $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

**(ii)** $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$

i. Let $\vec{b_1}$ and $\vec{b_2}$ be the vectors parallel to the pair of lines, $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$, respectively.

$\therefore \vec{b_1}=2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{b_2}=-\hat{i}+8\hat{j}+4\hat{k}$

$$|\vec{b_1}|=\sqrt{(2)^2+(5)^2+(-3)^2}=\sqrt{38}$$

$$|\vec{b_2}|=\sqrt{(-1)^2+(8)^2+(4)^2}=\sqrt{81}=9$$

$$\begin{aligned}\vec{b_1}\cdot\vec{b_2}&=\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\cdot\left(-\hat{i}+8\hat{j}+4\hat{k}\right)\\&=2(-1)+5\times 8+(-3)\cdot 4\\&=-2+40-12\\&=26\end{aligned}$$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

**(ii)** Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines, $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}$, respectively.

$$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$$

$$\therefore |\vec{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$$

$$|\vec{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$$

$$\vec{b}_1 \cdot \vec{b}_2 = \left(2\hat{i} + 2\hat{j} + \hat{k}\right) \cdot \left(4\hat{i} + \hat{j} + 8\hat{k}\right)$$

$$= 2\times4 + 2\times1 + 1\times8$$

$$= 8 + 2 + 8$$

$$= 18$$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$$\Rightarrow \cos Q = \frac{18}{3\times9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

**Question 12:**

Find the values of $p$ so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and

$\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles.

The given equations can be written in the standard form as

$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2}$ and $\frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$

The direction ratios of the lines are −3, $\frac{2p}{7}$, 2 and $\frac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, $a_1$, $b_1$, $c_1$ and $a_2$, $b_2$, $c_2$, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$$\therefore (-3)\cdot\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\cdot(1)+2\cdot(-5)=0$$
$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$
$$\Rightarrow 11p=70$$
$$\Rightarrow p=\frac{70}{11}$$

Thus, the value of $p$ is $\frac{70}{11}$.

**Question 13:**

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

The equations of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, $a_1$, $b_1$, $c_1$ and $a_2$, $b_2$, $c_2$, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$

$= 7 - 10 + 3$

$= 0$

Therefore, the given lines are perpendicular to each other.

**Question 14:**

Find the shortest distance between the lines

$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$ and
$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

The equations of the given lines are

$$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$$
$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right)\cdot\left(\vec{a}_2 - \vec{a}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right| \qquad \ldots(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$$
$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$$
$$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$
$$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$$
$$\vec{a}_2 - \vec{a}_1 = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{(-3\hat{i} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-3.1 + 3(-2)}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-9}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

**Question 15:**

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,
$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, is given by,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \qquad \ldots(1)$$

Comparing the given equations, we obtain

$$x_1 = -1,\ y_1 = -1,\ z_1 = -1$$
$$a_1 = 7,\ b_1 = -6,\ c_1 = 1$$
$$x_2 = 3,\ y_2 = 5,\ z_2 = 7$$
$$a_2 = 1,\ b_2 = -2,\ c_2 = 1$$

$$\text{Then,} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= 4(-6+2) - 6(7-1) + 8(-14+6)$$
$$= -16 - 36 - 64$$
$$= -116$$

$$\Rightarrow \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$$
$$= \sqrt{16+36+64}$$
$$= \sqrt{116}$$
$$= 2\sqrt{29}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2 \times 29}{\sqrt{29}} = -2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

**Question 16:**

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$
$$\text{and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

The given lines are $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left| \frac{\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|} \right| \qquad ...(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$$
$$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$$
$$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$
$$\vec{a}_2 - \vec{a}_1 = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) = \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right) \cdot \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right)$$
$$= -9 \times 3 + 3 \times 3 + 9 \times 3$$
$$= 9$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{9}{3\sqrt{19}} \right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

**Question 17:**

Find the shortest distance between the lines whose vector equations are

$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and
$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

The given lines are

$$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$$
$$\Rightarrow \vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}+\hat{j}-2\hat{k}\right) \qquad ...(1)$$

$$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$$
$$\Rightarrow \vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+s\left(\hat{i}+2\hat{j}-2\hat{k}\right) \qquad ...(2)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(3)$$

For the given equations,

$$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$$
$$\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$$
$$\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$$
$$\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}=(-2+4)\hat{i}-(2+2)\hat{j}+(-2-1)\hat{k}=2\hat{i}-4\hat{j}-3\hat{k}$$
$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{4+16+9}=\sqrt{29}$$

$\therefore \left(\vec{b}_1 \times \vec{b}_2\right)\cdot\left(\vec{a}_2 - \vec{a}_1\right) = \left(2\hat{i} - 4\hat{j} - 3\hat{k}\right)\cdot\left(\hat{j} - 4\hat{k}\right) = -4 + 12 = 8$

Substituting all the values in equation (3), we obtain

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

## Exercise-11.3

**Question 1:**

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a)z = 2 (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d)5*y* + 8 = 0

**(a)** The equation of the plane is $z = 2$ or $0x + 0y + z = 2$ … (1)

The direction ratios of normal are 0, 0, and 1.

$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$

Dividing both sides of equation (1) by 1, we obtain

$0.x + 0.y + 1.z = 2$

This is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

**(b)** $x + y + z = 1$ … (1)

The direction ratios of normal are 1, 1, and 1.

$\therefore \sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}} \quad ...(2)$$

This equation is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \text{ and } \frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

**(c)** $2x + 3y - z = 5$ … (1)

The direction ratios of normal are 2, 3, and −1.

$\therefore \sqrt{(2)^2+(3)^2+(-1)^2}=\sqrt{14}$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

$$\frac{2}{\sqrt{14}}x+\frac{3}{\sqrt{14}}y-\frac{1}{\sqrt{14}}z=\frac{5}{\sqrt{14}}$$

This equation is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \text{ and } \frac{-1}{\sqrt{14}}$ and the distance of normal from the origin is $\frac{5}{\sqrt{14}}$ units.

**(d)** $5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8$ … (1)

The direction ratios of normal are 0, −5, and 0.

$$\therefore \sqrt{0+(-5)^2+0}=5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y=\frac{8}{5}$$

This equation is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are 0, −1, and 0 and the distance of normal from the origin is $\frac{8}{5}$ units.

**Question 2:**

Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i}+5\hat{j}-6\hat{k}$.

The normal vector is, $\vec{n}=3\hat{i}+5\hat{j}-6\hat{k}$

$$\therefore \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{(3)^2+(5)^2+(6)^2}}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}$$

It is known that the equation of the plane with position vector $\vec{r}$ is given by, $\vec{r}\cdot\hat{n}=d$

$$\Rightarrow \hat{r}\cdot\left(\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\right)=7$$

This is the vector equation of the required plane.

**Question 3:**

Find the Cartesian equation of the following planes:

(a) $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})=2$ (b) $\vec{r}\cdot(2\hat{i}+3\hat{j}-4\hat{k})=1$

(c) $\vec{r}\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

**(a)** It is given that equation of the plane is

$\vec{r}\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2$ ...(1)

For any arbitrary point P ($x$, $y$, $z$) on the plane, position vector $\vec{r}$ is given by, $\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right)\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$
$$\Rightarrow x+y-z=2$$

This is the Cartesian equation of the plane.

**(b)** $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$ ...(1)

For any arbitrary point P ($x$, $y$, $z$) on the plane, position vector $\vec{r}$ is given by, $\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$$
$$\Rightarrow 2x+3y-4z=1$$

This is the Cartesian equation of the plane.

**(c)** $\vec{r}\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$ ...(1)

For any arbitrary point P ($x$, $y$, $z$) on the plane, position vector $\vec{r}$ is given by, $\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right)\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$$
$$\Rightarrow (s-2t)x+(3-t)y+(2s+t)z=15$$

This is the Cartesian equation of the given plane.

Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

**(a)** $2x+3y+4z-12=0$ **(b)** $3y+4z-6=0$

**(c)** $x+y+z=1$ **(d)** $5y+8=0$

**(a)** Let the coordinates of the foot of perpendicular P from the origin to the plane be $(x_1, y_1, z_1)$.

$2x + 3y + 4z - 12 = 0$

$\Rightarrow 2x + 3y + 4z = 12$ … (1)

The direction ratios of normal are 2, 3, and 4.

$$\therefore \sqrt{(2)^2+(3)^2+(4)^2}=\sqrt{29}$$

Dividing both sides of equation (1) by $\sqrt{29}$, we obtain

$$\frac{2}{\sqrt{29}}x+\frac{3}{\sqrt{29}}y+\frac{4}{\sqrt{29}}z=\frac{12}{\sqrt{29}}$$

This equation is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$(ld, md, nd)$.

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{2}{\sqrt{29}}.\frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}}.\frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}}.\frac{12}{\sqrt{29}}\right) \text{ i.e., } \left(\frac{24}{29}, \frac{36}{49}, \frac{48}{29}\right).$$

**(b)** Let the coordinates of the foot of perpendicular P from the origin to the plane be $(x_1, y_1, z_1)$.

$$3y + 4z - 6 = 0$$

$$\Rightarrow 0x + 3y + 4z = 6 \quad \ldots (1)$$

The direction ratios of the normal are 0, 3, and 4.

$$\therefore \sqrt{0 + 3^2 + 4^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$$

This equation is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$(ld, md, nd)$.

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, \frac{3}{5}.\frac{6}{5}, \frac{4}{5}.\frac{6}{5}\right) \text{ i.e., } \left(0, \frac{18}{25}, \frac{24}{25}\right).$$

**(c)** Let the coordinates of the foot of perpendicular P from the origin to the plane be $(x_1, y_1, z_1)$.

$$x + y + z = 1 \quad \ldots (1)$$

The direction ratios of the normal are 1, 1, and 1.

$$\therefore \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}}$$

This equation is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$(ld, md, nd)$.

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}\right) \text{ i.e., } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

**(d)** Let the coordinates of the foot of perpendicular P from the origin to the plane be $(x_1, y_1, z_1)$.

$$5y+8=0$$

$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$

The direction ratios of the normal are 0, −5, and 0.

$$\therefore \sqrt{0+(-5)^2+0}=5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y=\frac{8}{5}$$

This equation is of the form $lx + my + nz = d$, where $l$, $m$, $n$ are the direction cosines of normal to the plane and $d$ is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$(ld, md, nd)$.

Therefore, the coordinates of the foot of the perpendicular are

$\left(0, -1\left(\frac{8}{5}\right), 0\right)$ i.e., $\left(0, -\frac{8}{5}, 0\right)$.

**Question 5:**

Find the vector and Cartesian equation of the planes

(a) that passes through the point (1, 0, −2) and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$.

(b) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$.

**(a)** The position vector of point (1, 0, −2) is $\vec{a}=\hat{i}-2\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}+\hat{j}-\hat{k}$

The vector equation of the plane is given by, $(\vec{r}-\vec{a}).\vec{N}=0$

$$\Rightarrow\left[\vec{r}-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0 \qquad ...(1)$$

$\vec{r}$ is the position vector of any point P $(x, y, z)$ in the plane.

$\therefore \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Therefore, equation (1) becomes

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$
$$\Rightarrow\left[(x-1)\hat{i}+y\hat{j}+(z+2)\hat{k}\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$
$$\Rightarrow(x-1)+y-(z+2)=0$$
$$\Rightarrow x+y-z-3=0$$
$$\Rightarrow x+y-z=3$$

This is the Cartesian equation of the required plane.

**(b)** The position vector of the point (1, 4, 6) is $\vec{a}=\hat{i}+4\hat{j}+6\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N} = \hat{i} - 2\hat{j} + \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}).\vec{N} = 0$

$$\Rightarrow \left[\vec{r} - \left(\hat{i} + 4\hat{j} + 6\hat{k}\right)\right].\left(\hat{i} - 2\hat{j} + \hat{k}\right) = 0 \qquad ...(1)$$

$\vec{r}$ is the position vector of any point P ($x$, $y$, $z$) in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\left[\left(x\hat{i} + y\hat{j} + z\hat{k}\right) - \left(\hat{i} + 4\hat{j} + 6\hat{k}\right)\right].\left(\hat{i} - 2\hat{j} + \hat{k}\right) = 0$$
$$\Rightarrow \left[(x-1)\hat{i} + (y-4)\hat{j} + (z-6)\hat{k}\right]\cdot\left(\hat{i} - 2\hat{j} + \hat{k}\right) = 0$$
$$\Rightarrow (x-1) - 2(y-4) + (z-6) = 0$$
$$\Rightarrow x - 2y + z + 1 = 0$$

This is the Cartesian equation of the required plane.

Question 6:

Find the equations of the planes that passes through three points.

(a) (1, 1, −1), (6, 4, −5), (−4, −2, 3)

(b) (1, 1, 0), (1, 2, 1), (−2, 2, −1)

**(a)** The given points are A (1, 1, −1), B (6, 4, −5), and C (−4, −2, 3).

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12-10) - (18-20) - (-12+16)$$
$$= 2 + 2 - 4$$
$$= 0$$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

**(b)** The given points are A (1, 1, 0), B (1, 2, 1), and C (−2, 2, −1).

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2-2)-(2+2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

It is known that the equation of the plane through the points, $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z = 0$$

$$\Rightarrow -2x-3y+3z+2+3 = 0$$

$$\Rightarrow -2x-3y+3z = -5$$

$$\Rightarrow 2x+3y-3z = 5$$

This is the Cartesian equation of the required plane.

Question 7:

Find the intercepts cut off by the plane $2x+y-z=5$

$$2x+y-z=5 \qquad ...(1)$$

Dividing both sides of equation (1) by 5, we obtain

$$\frac{2}{5}x+\frac{y}{5}-\frac{z}{5}=1$$

$$\Rightarrow \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1 \qquad ...(2)$$

It is known that the equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$,
where *a*, *b*, *c* are the intercepts cut off by the plane at *x*, *y*, and *z* axes respectively.

Therefore, for the given equation,

$a=\frac{5}{2}, b=5, \text{ and } c=-5$

Thus, the intercepts cut off by the plane are $\frac{5}{2}$, 5, and $-5$.

**Question 8:**

Find the equation of the plane with intercept 3 on the *y*-axis and parallel to ZOX plane.

The equation of the plane ZOX is

$y = 0$

Any plane parallel to it is of the form, $y = a$

Since the *y*-intercept of the plane is 3,

$\therefore a = 3$

Thus, the equation of the required plane is $y = 3$

**Question 9:**

Find the equation of the plane through the intersection of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$ and the point (2, 2, 1)

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$, is

$$(3x-y+2z-4)+\alpha(x+y+z-2)=0, \text{ where } \alpha \in \mathrm{R} \quad \ldots(1)$$

The plane passes through the point (2, 2, 1). Therefore, this point will satisfy equation (1).

$\therefore (3\times 2-2+2\times 1-4)+\alpha(2+2+1-2)=0$

$\Rightarrow 2+3\alpha=0$

$\Rightarrow \alpha=-\frac{2}{3}$

Substituting $\alpha=-\frac{2}{3}$ in equation (1), we obtain

$(3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$

$\Rightarrow 3(3x-y+2z-4)-2(x+y+z-2)=0$

$\Rightarrow (9x-3y+6z-12)-2(x+y+z-2)=0$

$\Rightarrow 7x-5y+4z-8=0$

This is the required equation of the plane.

Question 10:

Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=7,\ \vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})=9$ and through the point (2, 1, 3)

The equations of the planes are $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=7$ and $\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$

$\Rightarrow \vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})-7=0 \quad ...(1)$

$\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})-9=0 \quad ...(2)$

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$\left[\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})-7\right]+\lambda\left[\vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})-9\right]=0$, where $\lambda\in R$

$\vec{r}\cdot\left[(2\hat{i}+2\hat{j}-3\hat{k})+\lambda(2\hat{i}+5\hat{j}+3\hat{k})\right]=9\lambda+7$

$\vec{r}\cdot\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7 \quad ...(3)$

The plane passes through the point (2, 1, 3). Therefore, its position vector is given by,

$\vec{r} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

Substituting in equation (3), we obtain

$$\left(2\hat{i} + \hat{j} - 3\hat{k}\right).\left[(2+2\lambda)\hat{i} + (2+5\lambda)\hat{j} + (3\lambda-3)\hat{k}\right] = 9\lambda + 7$$
$$\Rightarrow (2+2\lambda) + (2+5\lambda) + (3\lambda-3) = 9\lambda + 7$$
$$\Rightarrow 18\lambda - 3 = 9\lambda + 7$$
$$\Rightarrow 9\lambda = 10$$
$$\Rightarrow \lambda = \frac{10}{9}$$

Substituting $\lambda = \frac{10}{9}$ in equation (3), we obtain

$$\vec{r} \cdot \left(\frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k}\right) = 17$$
$$\Rightarrow \vec{r} \cdot \left(38\hat{i} + 68\hat{j} + 3\hat{k}\right) = 153$$

This is the vector equation of the required plane.

**Question 11:**

Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$

The equation of the plane through the intersection of the planes, $x + y + z = 1$ and $2x + 3y + 4z = 5$, is

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$
$$\Rightarrow (2\lambda+1)x + (3\lambda+1)y + (4\lambda+1)z - (5\lambda+1) = 0 \qquad ...(1)$$

The direction ratios, $a_1$, $b_1$, $c_1$, of this plane are $(2\lambda + 1)$, $(3\lambda + 1)$, and $(4\lambda + 1)$.

The plane in equation (1) is perpendicular to $x - y + z = 0$

Its direction ratios, $a_2$, $b_2$, $c_2$, are 1, −1, and 1.

Since the planes are perpendicular,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow (2\lambda+1) - (3\lambda+1) + (4\lambda+1) = 0$

$\Rightarrow 3\lambda + 1 = 0$

$\Rightarrow \lambda = -\frac{1}{3}$

Substituting $\lambda = -\frac{1}{3}$ in equation (1), we obtain

$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$

$\Rightarrow x - z + 2 = 0$

This is the required equation of the plane.

Question 12:

Find the angle between the planes whose vector equations are

$\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$.

The equations of the given planes are $\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$

It is known that if $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, then the angle between them, Q, is given by,

$$\cos Q = \left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right| \qquad \text{...(1)}$$

Here, $\vec{n}_1 = 2\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{n}_2 = 3\hat{i}-3\hat{j}+5\hat{k}$

$\therefore \vec{n}_1\cdot\vec{n}_2 = \left(2\hat{i}+2\hat{j}-3\hat{k}\right)\left(3\hat{i}-3\hat{j}+5\hat{k}\right) = 2.3+2.(-3)+(-3).5 = -15$

$|\vec{n}_1| = \sqrt{(2)^2+(2)^2+(-3)^2} = \sqrt{17}$

$|\vec{n}_2| = \sqrt{(3)^2+(-3)^2+(5)^2} = \sqrt{43}$

Substituting the value of $\vec{n}\cdot\vec{n}_2$, $|\vec{n}_1|$ and $|\vec{n}_2|$ in equation (1), we obtain

$\cos Q = \left| \frac{-15}{\sqrt{17} \cdot \sqrt{43}} \right|$

$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$

$\Rightarrow \cos Q^{-1} = \left( \frac{15}{\sqrt{731}} \right)$

**Question 13:**

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

The direction ratios of normal to the plane, $L_1 : a_1x + b_1y + c_1z = 0$, are $a_1$, $b_1$, $c_1$ and $L_2 : a_1x + b_2y + c_2z = 0$ are $a_2$, $b_2$, $c_2$.

$L_1 \parallel L_2$, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$L_1 \perp L_2$, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The angle between $L_1$ and $L_2$ is given by,

$$Q = \cos^{-1} \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

**(a)** The equations of the planes are $7x + 5y + 6z + 30 = 0$ and

$3x - y - 10z + 4 = 0$

Here, $a_1 = 7$, $b_1 = 5$, $c_1 = 6$

$a_2 = 3,\ b_2 = -1, c_2 = -10$

$a_1a_2 + b_1b_2 + c_1c_2 = 7\times 3 + 5\times(-1) + 6\times(-10) = -44 \neq 0$

Therefore, the given planes are not perpendicular.

$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$

It can be seen that, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$Q = \cos^{-1}\left|\frac{7\times 3 + 5\times(-1) + 6\times(-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2}\times\sqrt{(3)^2 + (-1)^2 + (-10)^2}}\right|$$

$$= \cos^{-1}\left|\frac{21-5-60}{\sqrt{110}\times\sqrt{110}}\right|$$

$$= \cos^{-1}\frac{44}{110}$$

$$= \cos^{-1}\frac{2}{5}$$

**(b)** The equations of the planes are $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

Here, $a_1 = 2,\ b_1 = 1,\ c_1 = 3$ and $a_2 = 1,\ b_2 = -2, c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2\times 1 + 1\times(-2) + 3\times 0 = 0$

Thus, the given planes are perpendicular to each other.

**(c)** The equations of the given planes are $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

Here, $a_1 = 2, b_1 - 2, c_1 = 4$ and $a_2 = 3, b_2 = -3, c_2 = 6$

$$a_1a_2 + b_1b_2 + c_1c_2 = 2\times3 + (-2)(-3) + 4\times6 = 6+6+24 = 36 \neq 0$$

Thus, the given planes are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{and} \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given planes are parallel to each other.

**(d)** The equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

Here, $a_1 = 2, b_1 = -1, c_1 = 3$ and $a_2 = 2, b_2 = -1, c_2 = 3$

$$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{and } \frac{c_1}{c_2} = \frac{3}{3} = 1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given lines are parallel to each other.

**(e)** The equations of the given planes are $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Here, $a_1 = 4, b_1 = 8, c_1 = 1$ and $a_2 = 0, b_2 = 1, c_2 = 1$

$$a_1a_2 + b_1b_2 + c_1c_2 = 4\times0 + 8\times1 + 1 = 9 \neq 0$$

Therefore, the given lines are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{4}{0}, \ \frac{b_1}{b_2} = \frac{8}{1} = 8, \ \frac{c_1}{c_2} = \frac{1}{1} = 1$$

$$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q = \cos^{-1}\left|\frac{4\times 0+8\times 1+1\times 1}{\sqrt{4^2+8^2+1^2}\times\sqrt{0^2+1^2+1^2}}\right| = \cos^{-1}\left|\frac{9}{9\times\sqrt{2}}\right| = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

**Point Plane**

(a) (0, 0, 0) $3x-4y+12z=3$

(b) (3, −2, 1) $2x-y+2z+3=0$

(c) (2, 3, −5) $x+2y-2z=9$

(d) (−6, 0, 0) $2x-3y+6z-2=0$

It is known that the distance between a point, $p(x_1, y_1, z_1)$, and a plane, $Ax + By + Cz = D$, is given by,

$$d = \left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right| \qquad ...(1)$$

**(a)** The given point is (0, 0, 0) and the plane is $3x-4y+12z=3$

$$\therefore d = \left|\frac{3\times 0-4\times 0+12\times 0-3}{\sqrt{(3)^2+(-4)^2+(12)^2}}\right| = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

**(b)** The given point is (3, − 2, 1) and the plane is $2x-y+2z+3=0$

$$\therefore d = \left|\frac{2\times 3-(-2)+2\times 1+3}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right| = \left|\frac{13}{3}\right| = \frac{13}{3}$$

**(c)** The given point is (2, 3, −5) and the plane is $x+2y-2z=9$

$$\therefore d=\left|\frac{2+2\times 3-2(-5)-9}{\sqrt{(1)^2+(2)^2+(-2)^2}}\right|=\frac{9}{3}=3$$

**(d)** The given point is (−6, 0, 0) and the plane is $2x-3y+6z-2=0$

$$d=\left|\frac{2(-6)-3\times 0+6\times 0-2}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right|=\left|\frac{-14}{\sqrt{49}}\right|=\frac{14}{7}=2$$