Question 1:

Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer :

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by
$T_{r+1} = {}^nC_r a^{n-r} b^r$.

The first three terms of the expansion are given as 729, 7290, and 30375 respectively.

Therefore, we obtain

$$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \quad ...(1)$$
$$T_2 = {}^nC_1 a^{n-1} b^1 = na^{n-1}b = 7290 \quad ...(2)$$
$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \quad ...(3)$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$
$$\Rightarrow \frac{nb}{a} = 10 \quad ...(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$
$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$
$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{25}{3}$$
$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$
$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \quad \left[\text{Using } (4)\right]$$
$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \quad ...(5)$$

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$
$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain

$a^6 = 729$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Answer :

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that x^2 occurs in the $(r + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{r+1} = {}^9C_r (3)^{9-r} (ax)^r = {}^9C_r (3)^{9-r} a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r = 2$

Thus, the coefficient of x^2 is

$${}^9C_2 (3)^{9-2} a^2 = \frac{9!}{2!7!} (3)^7 a^2 = 36 (3)^7 a^2$$

Assuming that x^3 occurs in the $(k + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{k+1} = {}^9C_k (3)^{9-k} (ax)^k = {}^9C_k (3)^{9-k} a^k x^k$$

Comparing the indices of x in x^3 and in T_{k+1}, we obtain

$k = 3$

Thus, the coefficient of x^3 is

$${}^9C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!}(3)^6 a^3 = 84(3)^6 a^3$$

It is given that the coefficients of x^2 and x^3 are the same.

$$84(3)^6 a^3 = 36(3)^7 a^2$$
$$\Rightarrow 84a = 36 \times 3$$
$$\Rightarrow a = \frac{36 \times 3}{84} = \frac{104}{84}$$
$$\Rightarrow a = \frac{9}{7}$$

Thus, the required value of a is $\frac{9}{7}$.

Question 3:

Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer :

Using Binomial Theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$$\begin{aligned}(1+2x)^6 &= {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 \\ &\quad + {}^6C_5(2x)^5 + {}^6C_6(2x)^6 \\ &= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\end{aligned}$$

$$
\begin{aligned}
(1-x)^7 &= {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 \\
&\quad - {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7 \\
&= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7
\end{aligned}
$$

$$
\begin{aligned}
&\therefore (1+2x)^6(1-x)^7 \\
&= \left(1+12x+60x^2+160x^3+240x^4+192x^5+64x^6\right)\left(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7\right)
\end{aligned}
$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5, are required.

The terms containing x^5 are

$$
\begin{aligned}
&1(-21x^5)+(12x)(35x^4)+(60x^2)(-35x^3)+(160x^3)(21x^2)+(240x^4)(-7x)+(192x^5)(1) \\
&= 171x^5
\end{aligned}
$$

Thus, the coefficient of x^5 in the given product is 171.

Question 4:

If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[**Hint:** write $a^n = (a - b + b)^n$ and expand]

Answer :

In order to prove that $(a - b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k\,(a - b)$, where k is some natural number

It can be written that, $a = a - b + b$

$$\therefore a^n = (a-b+b)^n = \left[(a-b)+b\right]^n$$
$$= {}^nC_0 (a-b)^n + {}^nC_1 (a-b)^{n-1} b + \ldots + {}^nC_{n-1} (a-b) b^{n-1} + {}^nC_n b^n$$
$$= (a-b)^n + {}^nC_1 (a-b)^{n-1} b + \ldots + {}^nC_{n-1} (a-b) b^{n-1} + b^n$$
$$\Rightarrow a^n - b^n = (a-b)\left[(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + \ldots + {}^nC_{n-1} b^{n-1}\right]$$
$$\Rightarrow a^n - b^n = k(a-b)$$
where, $k = \left[(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + \ldots + {}^nC_{n-1} b^{n-1}\right]$ is a natural number

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.

Answer :

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem.

This can be done as

$$(a+b)^6 = {}^6C_0 a^6 + {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 + {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 + {}^6C_5 a^1 b^5 + {}^6C_6 b^6$$
$$= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$$
$$(a-b)^6 = {}^6C_0 a^6 - {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 - {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 - {}^6C_5 a^1 b^5 + {}^6C_6 b^6$$
$$= a^6 - 6a^5 b + 15a^4 b^2 - 20a^3 b^3 + 15a^2 b^4 - 6ab^5 + b^6$$
$$\therefore (a+b)^6 - (a-b)^6 = 2\left[6a^5 b + 20a^3 b^3 + 6ab^5\right]$$
Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain
$$\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6 = 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right) + 20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + 6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$$
$$= 2\left[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}\right]$$
$$= 2 \times 198\sqrt{6}$$
$$= 396\sqrt{6}$$

Question 6:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$.

Answer :

Firstly, the expression $(x+y)^4+(x-y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$\begin{aligned}
(x+y)^4 &= {}^4C_0x^4+{}^4C_1x^3y+{}^4C_2x^2y^2+{}^4C_3xy^3+{}^4C_4y^4\\
&= x^4+4x^3y+6x^2y^2+4xy^3+y^4\\
(x-y)^4 &= {}^4C_0x^4-{}^4C_1x^3y+{}^4C_2x^2y^2-{}^4C_3xy^3+{}^4C_4y^4\\
&= x^4-4x^3y+6x^2y^2-4xy^3+y^4
\end{aligned}$$

$$\therefore (x+y)^4+(x-y)^4=2\left(x^4+6x^2y^2+y^4\right)$$

Putting $x=a^2$ and $y=\sqrt{a^2-1}$, we obtain

$$\begin{aligned}
\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4 &= 2\left[\left(a^2\right)^4+6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2+\left(\sqrt{a^2-1}\right)^4\right]\\
&= 2\left[a^8+6a^4\left(a^2-1\right)+\left(a^2-1\right)^2\right]\\
&= 2\left[a^8+6a^6-6a^4+a^4-2a^2+1\right]\\
&= 2\left[a^8+6a^6-5a^4-2a^2+1\right]\\
&= 2a^8+12a^6-10a^4-4a^2+2
\end{aligned}$$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer :

$0.99 = 1 - 0.01$

$$\begin{aligned}\therefore (0.99)^5 &= (1-0.01)^5 \\ &= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \qquad \text{(Approximately)} \\ &= 1-5(0.01)+10(0.01)^2 \\ &= 1-0.05+0.001 \\ &= 1.001-0.05 \\ &= 0.951\end{aligned}$$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 8:

Find *n*, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$

Answer :

In the expansion, $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$,

Fifth term from the beginning $= {}^nC_4a^{n-4}b^4$

Fifth term from the end $= {}^nC_{n-4}a^4b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$, the fifth term from the beginning is ${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$.

$${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3} = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{2}\cdot\frac{1}{3} = \frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n \qquad \ldots(1)$$

$${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4}.2\cdot\frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n} = {}^nC_{n-4}.2\cdot\frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} \qquad \ldots(2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}:1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} \times \frac{\left(\sqrt[4]{3}\right)^n}{6} = \sqrt{6}$$

$$\Rightarrow \left(\sqrt[4]{6}\right)^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4\times\frac{5}{2} = 10$$

Thus, the value of *n* is 10.

Question 9:

Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$.

Answer :

Using Binomial Theorem, the given expression $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$ can be expanded as

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$=\left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \qquad \ldots(1)$$

Again by using Binomial Theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$= 1 + 4\times\frac{x}{2} + 6\times\frac{x^2}{4} + 4\times\frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \qquad \ldots(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

Question 10:

Find the expansion of $\left(3x^2-2ax+3a^2\right)^3$ using binomial theorem.

Answer :

Using Binomial Theorem, the given expression $\left(3x^2-2ax+3a^2\right)^3$ can be expanded as

$$\left[\left(3x^2-2ax\right)+3a^2\right]^3$$
$$={}^3C_0\left(3x^2-2ax\right)^3+{}^3C_1\left(3x^2-2ax\right)^2\left(3a^2\right)+{}^3C_2\left(3x^2-2ax\right)\left(3a^2\right)^2+{}^3C_3\left(3a^2\right)^3$$
$$=\left(3x^2-2ax\right)^3+3\left(9x^4-12ax^3+4a^2x^2\right)\left(3a^2\right)+3\left(3x^2-2ax\right)\left(9a^4\right)+27a^6$$
$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+36a^4x^2+81a^4x^2-54a^5x+27a^6$$
$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6 \qquad \ldots(1)$$

Again by using Binomial Theorem, we obtain

$$\left(3x^2-2ax\right)^3$$
$$={}^3C_0\left(3x^2\right)^3-{}^3C_1\left(3x^2\right)^2\left(2ax\right)+{}^3C_2\left(3x^2\right)\left(2ax\right)^2-{}^3C_3\left(2ax\right)^3$$
$$=27x^6-3\left(9x^4\right)\left(2ax\right)+3\left(3x^2\right)\left(4a^2x^2\right)-8a^3x^3$$
$$=27x^6-54ax^5+36a^2x^4-8a^3x^3 \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\left(3x^2-2ax+3a^2\right)^3$$
$$=27x^6-54ax^5+36a^2x^4-8a^3x^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6$$
$$=27x^6-54ax^5+117a^2x^4-116a^3x^3+117a^4x^2-54a^5x+27a^6$$