

Chapter-5 Complex Numbers & Quadratic Equations

Miscellaneous

Question 1:

Evaluate: $\left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3$

Answer :

$$\begin{aligned}
 & \left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3 \\
 &= \left[i^{4 \times 4 + 2} + \frac{1}{i^{4 \times 6 + 1}} \right]^3 \\
 &= \left[(i^4)^4 \cdot i^2 + \frac{1}{(i^4)^6 \cdot i} \right]^3 \\
 &= \left[i^2 + \frac{1}{i} \right]^3 & [i^4 = 1] \\
 &= \left[-1 + \frac{1}{i} \times \frac{i}{i} \right]^3 & [i^2 = -1] \\
 &= \left[-1 + \frac{i}{i^2} \right]^3 \\
 &= [-1 - i]^3 \\
 &= (-1)^3 [1 + i]^3 \\
 &= -[1^3 + i^3 + 3 \cdot 1 \cdot i(1 + i)] \\
 &= -[1 + i^3 + 3i + 3i^2] \\
 &= -[1 - i + 3i - 3] \\
 &= -[-2 + 2i] \\
 &= 2 - 2i
 \end{aligned}$$

Question 2:

For any two complex numbers z_1 and z_2 , prove that

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Answer :

$$\begin{aligned} \text{Let } z_1 &= x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2 \\ \therefore z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2 \quad [i^2 = -1] \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \\ \Rightarrow \operatorname{Re}(z_1 z_2) &= x_1 x_2 - y_1 y_2 \\ \Rightarrow \operatorname{Re}(z_1 z_2) &= \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2 \\ \text{Hence, proved.} \end{aligned}$$

Question 3:

Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Answer :

$$\begin{aligned} \left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) &= \left[\frac{(1+i) - 2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right] \\ &= \left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right] = \left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right] \\ &= \left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right] = \frac{33+31i}{28-10i} = \frac{33+31i}{2(14-5i)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(33+31i)(14+5i)}{2(14-5i)(14+5i)} \quad [\text{On multiplying numerator and denominator by } (14+5i)] \\
 &= \frac{462+165i+434i+155i^2}{2[(14)^2-(5i)^2]} = \frac{307+599i}{2(196-25i^2)} \\
 &= \frac{307+599i}{2(221)} = \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}
 \end{aligned}$$

This is the required standard form.

Question 4:

If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$.

Answer :

$$\begin{aligned}
 x - iy &= \sqrt{\frac{a-ib}{c-id}} \\
 &= \sqrt{\frac{a-ib}{c-id} \times \frac{c+id}{c+id}} \quad [\text{On multiplying numerator and denominator by } (c+id)] \\
 &= \sqrt{\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}} \\
 \therefore (x-iy)^2 &= \frac{(ac+bd)+i(ad-bc)}{c^2+d^2} \\
 \Rightarrow x^2 - y^2 - 2ixy &= \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}
 \end{aligned}$$

On comparing real and imaginary parts, we obtain

$$x^2 - y^2 = \frac{ac+bd}{c^2+d^2}, \quad -2xy = \frac{ad-bc}{c^2+d^2} \quad (1)$$

$$\begin{aligned}
 (x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 \\
 &= \left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{ad - bc}{c^2 + d^2}\right)^2 \quad [\text{Using (1)}] \\
 &= \frac{a^2c^2 + b^2d^2 + 2acbd + a^2d^2 + b^2c^2 - 2adbc}{(c^2 + d^2)^2} \\
 &= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{(c^2 + d^2)^2} \\
 &= \frac{a^2(c^2 + d^2) + b^2(c^2 + d^2)}{(c^2 + d^2)^2} \\
 &= \frac{(c^2 + d^2)(a^2 + b^2)}{(c^2 + d^2)^2} \\
 &= \frac{a^2 + b^2}{c^2 + d^2}
 \end{aligned}$$

Hence, proved.

Question 5:

Convert the following in the polar form:

$$(i) \frac{1+7i}{(2-i)^2}, (ii) \frac{1+3i}{1-2i}$$

Answer :

$$\begin{aligned}
 (i) \text{ Here, } z &= \frac{1+7i}{(2-i)^2} \\
 &= \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i} \\
 &= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{3^2+4^2} \\
 &= \frac{3+4i+21i-28}{3^2+4^2} = \frac{-25+25i}{25} \\
 &= -1+i
 \end{aligned}$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we obtain

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 2$$

$$\Rightarrow r^2 = 2 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$\Rightarrow r = \sqrt{2} \quad [\text{Conventionally, } r > 0]$$

$$\therefore \sqrt{2} \cos \theta = -1 \text{ and } \sqrt{2} \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in II quadrant}]$$

$$\therefore z = r \cos \theta + i r \sin \theta$$

$$= \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

This is the required polar form.

(ii) Here, $z = \frac{1+3i}{1-2i}$

$$= \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{1+2i+3i-6}{1+4}$$

$$= \frac{-5+5i}{5} = -1+i$$

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we obtain

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = 2$$

$$\Rightarrow r^2 = 2 \quad [\cos^2 \theta + \sin^2 \theta = 1]$$

$$\begin{aligned}\Rightarrow r &= \sqrt{2} && [\text{Conventionally, } r > 0] \\ \therefore \sqrt{2} \cos \theta &= -1 \text{ and } \sqrt{2} \sin \theta = 1 \\ \Rightarrow \cos \theta &= \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}} \\ \therefore \theta &= \pi - \frac{\pi}{4} = \frac{3\pi}{4} && [\text{As } \theta \text{ lies in II quadrant}]\end{aligned}$$

$$\begin{aligned}\therefore z &= r \cos \theta + i r \sin \theta \\ &= \sqrt{2} \cos \frac{3\pi}{4} + i \sqrt{2} \sin \frac{3\pi}{4} = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)\end{aligned}$$

This is the required polar form.

Question 6:

Solve the equation $3x^2 - 4x + \frac{20}{3} = 0$

Answer :

The given quadratic equation is $3x^2 - 4x + \frac{20}{3} = 0$

This equation can also be written as $9x^2 - 12x + 20 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = 9$, $b = -12$, and $c = 20$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$$

Therefore, the required solutions are

$$\begin{aligned}\frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-12) \pm \sqrt{-576}}{2 \times 9} = \frac{12 \pm \sqrt{576}i}{18} \quad [\sqrt{-1} = i] \\ &= \frac{12 \pm 24i}{18} = \frac{6(2 \pm 4i)}{18} = \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3}i\end{aligned}$$

Question 7:

Solve the equation $x^2 - 2x + \frac{3}{2} = 0$

Answer :

The given quadratic equation is $x^2 - 2x + \frac{3}{2} = 0$

This equation can also be written as $2x^2 - 4x + 3 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = 2$, $b = -4$, and $c = 3$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-4)^2 - 4 \times 2 \times 3 = 16 - 24 = -8$$

Therefore, the required solutions are

$$\begin{aligned}\frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}i}{4} \quad [\sqrt{-1} = i] \\ &= \frac{2 \pm \sqrt{2}i}{2} = 1 \pm \frac{\sqrt{2}}{2}i\end{aligned}$$

Question 8:

Solve the equation $27x^2 - 10x + 1 = 0$

Answer :

The given quadratic equation is $27x^2 - 10x + 1 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$$a = 27, b = -10, \text{ and } c = 1$$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8$$

Therefore, the required solutions are

$$\begin{aligned} \frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}i}{54} \quad [\sqrt{-1} = i] \\ &= \frac{5 \pm \sqrt{2}i}{27} = \frac{5}{27} \pm \frac{\sqrt{2}}{27}i \end{aligned}$$

Question 9:

Solve the equation $21x^2 - 28x + 10 = 0$

Answer :

The given quadratic equation is $21x^2 - 28x + 10 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$, we obtain

$$a = 21, b = -28, \text{ and } c = 10$$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10 = 784 - 840 = -56$$

Therefore, the required solutions are

$$\begin{aligned} \frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}i}{42} \\ &= \frac{28 \pm 2\sqrt{14}i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42}i = \frac{2}{3} \pm \frac{\sqrt{14}}{21}i \end{aligned}$$

Question 10:

If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$.

Answer :

$$\begin{aligned}
 z_1 &= 2-i, \quad z_2 = 1+i \\
 \therefore \left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right| &= \left| \frac{(2-i) + (1+i) + 1}{(2-i) - (1+i) + 1} \right| \\
 &= \left| \frac{4}{2-2i} \right| = \left| \frac{4}{2(1-i)} \right| \\
 &= \left| \frac{2}{1-i} \times \frac{1+i}{1+i} \right| = \left| \frac{2(1+i)}{1^2 - i^2} \right| \\
 &= \left| \frac{2(1+i)}{1+1} \right| \quad [i^2 = -1] \\
 &= \left| \frac{2(1+i)}{2} \right| \\
 &= |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}
 \end{aligned}$$

Thus, the value of $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$ is $\sqrt{2}$.

Question 11:

If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$

Answer :

$$\begin{aligned}
 a + ib &= \frac{(x+i)^2}{2x^2+1} \\
 &= \frac{x^2 + i^2 + 2xi}{2x^2+1} \\
 &= \frac{x^2 - 1 + i2x}{2x^2+1} \\
 &= \frac{x^2-1}{2x^2+1} + i \left(\frac{2x}{2x^2+1} \right)
 \end{aligned}$$

On comparing real and imaginary parts, we obtain

$$\begin{aligned}
 a &= \frac{x^2 - 1}{2x^2 + 1} \text{ and } b = \frac{2x}{2x^2 + 1} \\
 \therefore a^2 + b^2 &= \left(\frac{x^2 - 1}{2x^2 + 1} \right)^2 + \left(\frac{2x}{2x^2 + 1} \right)^2 \\
 &= \frac{x^4 + 1 - 2x^2 + 4x^2}{(2x^2 + 1)^2} \\
 &= \frac{x^4 + 1 + 2x^2}{(2x^2 + 1)^2} \\
 &= \frac{(x^2 + 1)^2}{(2x^2 + 1)^2} \\
 \therefore a^2 + b^2 &= \frac{(x^2 + 1)^2}{(2x^2 + 1)^2}
 \end{aligned}$$

Hence, proved.

Question 12:

Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

$$(i) \operatorname{Re} \left(\frac{z_1 z_2}{\bar{z}_1} \right), (ii) \operatorname{Im} \left(\frac{1}{z_1 \bar{z}_1} \right)$$

Answer :

$$z_1 = 2 - i, z_2 = -2 + i$$

$$(i) z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$$

$$\bar{z}_1 = 2 + i$$

$$\therefore \frac{z_1 z_2}{\bar{z}_1} = \frac{-3 + 4i}{2 + i}$$

On multiplying numerator and denominator by $(2 - i)$, we obtain

$$\begin{aligned}\frac{z_1 z_2}{\bar{z}_1} &= \frac{(-3+4i)(2-i)}{(2+i)(2-i)} = \frac{-6+3i+8i-4i^2}{2^2+1^2} = \frac{-6+11i-4(-1)}{2^2+1^2} \\ &= \frac{-2+11i}{5} = \frac{-2}{5} + \frac{11}{5}i\end{aligned}$$

On comparing real parts, we obtain

$$\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$$

$$(ii) \quad \frac{1}{z_1 \bar{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{(2)^2 + (1)^2} = \frac{1}{5}$$

On comparing imaginary parts, we obtain

$$\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right) = 0$$

Question 13:

Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$.

Answer :

Let $z = \frac{1+2i}{1-3i}$, then

$$\begin{aligned}z &= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2+3^2} = \frac{1+5i+6(-1)}{1+9} \\ &= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i\end{aligned}$$

Let $z = r \cos \theta + ir \sin \theta$

$$\text{i.e., } r \cos \theta = \frac{-1}{2} \text{ and } r \sin \theta = \frac{1}{2}$$

On squaring and adding, we obtain

$$\begin{aligned}
 r^2 (\cos^2 \theta + \sin^2 \theta) &= \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\
 \Rightarrow r^2 &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\
 \Rightarrow r &= \frac{1}{\sqrt{2}} \quad [\text{Conventionally, } r > 0] \\
 \therefore \frac{1}{\sqrt{2}} \cos \theta &= \frac{-1}{2} \text{ and } \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{2} \\
 \Rightarrow \cos \theta &= \frac{-1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}} \\
 \therefore \theta &= \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad [\text{As } \theta \text{ lies in the II quadrant}]
 \end{aligned}$$

Therefore, the modulus and argument of the given complex number are $\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$ respectively.

Question 14:

Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Answer :

$$\text{Let } z = (x - iy)(3 + 5i)$$

$$\begin{aligned}
 z &= 3x + 5xi - 3yi - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y) \\
 \therefore \bar{z} &= (3x + 5y) - i(5x - 3y)
 \end{aligned}$$

It is given that, $\bar{z} = -6 - 24i$

$$\therefore (3x + 5y) - i(5x - 3y) = -6 - 24i$$

Equating real and imaginary parts, we obtain

$$3x + 5y = -6 \quad \dots \text{ (i)}$$

$$5x - 3y = 24 \quad \dots \text{ (ii)}$$

Multiplying equation (i) by 3 and equation (ii) by 5 and then adding them, we obtain

$$\begin{aligned}
 9x + 15y &= -18 \\
 25x - 15y &= 120 \\
 \hline
 34x &= 102 \\
 \therefore x &= \frac{102}{34} = 3
 \end{aligned}$$

Putting the value of x in equation (i), we obtain

$$\begin{aligned}
 3(3) + 5y &= -6 \\
 \Rightarrow 5y &= -6 - 9 = -15 \\
 \Rightarrow y &= -3
 \end{aligned}$$

Thus, the values of x and y are 3 and -3 respectively.

Question 15:

Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

Answer :

$$\begin{aligned}
 \frac{1+i}{1-i} - \frac{1-i}{1+i} &= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\
 &= \frac{1+i^2+2i-1-i^2+2i}{1^2+1^2} \\
 &= \frac{4i}{2} = 2i \\
 \therefore \left| \frac{1+i}{1-i} - \frac{1-i}{1+i} \right| &= |2i| = \sqrt{2^2} = 2
 \end{aligned}$$

Question 16:

If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$.

Answer :

$$\begin{aligned}
(x + iy)^3 &= u + iv \\
\Rightarrow x^3 + (iy)^3 + 3 \cdot x \cdot iy(x + iy) &= u + iv \\
\Rightarrow x^3 + i^3 y^3 + 3x^2 yi + 3xy^2 i^2 &= u + iv \\
\Rightarrow x^3 - iy^3 + 3x^2 yi - 3xy^2 &= u + iv \\
\Rightarrow (x^3 - 3xy^2) + i(3x^2 y - y^3) &= u + iv
\end{aligned}$$

On equating real and imaginary parts, we obtain

$$\begin{aligned}
u &= x^3 - 3xy^2, \quad v = 3x^2 y - y^3 \\
\therefore \frac{u}{x} + \frac{v}{y} &= \frac{x^3 - 3xy^2}{x} + \frac{3x^2 y - y^3}{y} \\
&= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y} \\
&= x^2 - 3y^2 + 3x^2 - y^2 \\
&= 4x^2 - 4y^2 \\
&= 4(x^2 - y^2) \\
\therefore \frac{u}{x} + \frac{v}{y} &= 4(x^2 - y^2)
\end{aligned}$$

Hence, proved.

Question 17:

If α and β are different complex numbers with $|\beta| = 1$, then find $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right|$.

Answer :

Let $\alpha = a + ib$ and $\beta = x + iy$

It is given that, $|\beta| = 1$

$$\begin{aligned}
\therefore \sqrt{x^2 + y^2} &= 1 \\
\Rightarrow x^2 + y^2 &= 1 \quad \dots (i)
\end{aligned}$$

$$\begin{aligned}
\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| &= \left| \frac{(x + iy) - (a + ib)}{1 - (a - ib)(x + iy)} \right| \\
&= \left| \frac{(x - a) + i(y - b)}{1 - (ax + aiy - ibx + by)} \right| \\
&= \left| \frac{(x - a) + i(y - b)}{(1 - ax - by) + i(bx - ay)} \right| \\
&= \frac{|(x - a) + i(y - b)|}{|(1 - ax - by) + i(bx - ay)|} \quad \left[\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right] \\
&= \frac{\sqrt{(x - a)^2 + (y - b)^2}}{\sqrt{(1 - ax - by)^2 + (bx - ay)^2}} \\
&= \frac{\sqrt{x^2 + a^2 - 2ax + y^2 + b^2 - 2by}}{\sqrt{1 + a^2x^2 + b^2y^2 - 2ax + 2abxy - 2by + b^2x^2 + a^2y^2 - 2abxy}} \\
&= \frac{\sqrt{(x^2 + y^2) + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2(x^2 + y^2) + b^2(y^2 + x^2) - 2ax - 2by}} \\
&= \frac{\sqrt{1 + a^2 + b^2 - 2ax - 2by}}{\sqrt{1 + a^2 + b^2 - 2ax - 2by}} \quad [\text{Using (1)}] \\
&= 1 \\
\therefore \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| &= 1
\end{aligned}$$

Question 18:

Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.

Answer :

$$\begin{aligned}
|1-i|^x &= 2^x \\
\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x &= 2^x \\
\Rightarrow (\sqrt{2})^x &= 2^x \\
\Rightarrow 2^{\frac{x}{2}} &= 2^x \\
\Rightarrow \frac{x}{2} &= x \\
\Rightarrow x &= 2x \\
\Rightarrow 2x - x &= 0 \\
\Rightarrow x &= 0
\end{aligned}$$

Thus, 0 is the only integral solution of the given equation. Therefore, the number of non-zero integral solutions of the given equation is 0

Question 19:

If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, then show that

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2.$$

Answer :

$$(a+ib)(c+id)(e+if)(g+ih) = A+iB$$

$$\begin{aligned}
\therefore |(a+ib)(c+id)(e+if)(g+ih)| &= |A+iB| \\
\Rightarrow |(a+ib)| \times |(c+id)| \times |(e+if)| \times |(g+ih)| &= |A+iB| \quad [|z_1 z_2| = |z_1| |z_2|] \\
\Rightarrow \sqrt{a^2+b^2} \times \sqrt{c^2+d^2} \times \sqrt{e^2+f^2} \times \sqrt{g^2+h^2} &= \sqrt{A^2+B^2}
\end{aligned}$$

On squaring both sides, we obtain

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$$

Hence, proved.

Question 20:

If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m .

Answer :

$$\left(\frac{1+i}{1-i}\right)^m = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\Rightarrow \left(\frac{(1+i)^2}{1^2+1^2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1^2+i^2+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{1-1+2i}{2}\right)^m = 1$$

$$\Rightarrow \left(\frac{2i}{2}\right)^m = 1$$

$$\Rightarrow i^m = 1$$

$\therefore m = 4k$, where k is some integer.

Therefore, the least positive integer is 1.

Thus, the least positive integral value of m is 4 ($= 4 \times 1$).