Chapter-13 Limits & Derivatives

Miscellaneous

Question 1:

Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x + 1)$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Answer :

(i) Let $f(x) = -x$. Accordingly, $f(x+h) = -(x+h)$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{-(x+h)-(-x)}{h} \\
&= \lim_{h\to 0}\frac{-x-h+x}{h} \\
&= \lim_{h\to 0}\frac{-h}{h} \\
&= \lim_{h\to 0}(-1) = -1
\end{aligned}$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{x(x+h)} \\
&= \frac{1}{x\cdot x}=\frac{1}{x^2}
\end{aligned}
$$

(iii) Let $f(x) = \sin(x + 1)$. Accordingly, $f(x+h)=\sin(x+h+1)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right] \\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right] \\
&= \lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} && \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right] \\
&= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1 && \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right] \\
&= \cos(x+1)
\end{aligned}
$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]
\end{aligned}
$$

Question 2:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): ($x + a$)

Answer :

Let $f(x) = x + a$. Accordingly, $f(x+h)=x+h+a$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{x+h+a-x-a}{h} \\
&= \lim_{h\to 0}\left(\frac{h}{h}\right) \\
&= \lim_{h\to 0}(1) \\
&= 1
\end{aligned}
$$

Question 3:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(px+q)\left(\frac{r}{x}+s\right)$

Answer :

Let $f(x) = (px+q)\left(\frac{r}{x}+s\right)$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)' \\
&= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p) \\
&= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p \\
&= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p \\
&= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\
&= ps - \frac{qr}{x^2}
\end{aligned}
$$

Question 4:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)(cx + d)^2$

Answer :

Let $f(x) = (ax+b)(cx+d)^2$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= (ax+b)\frac{d}{dx}(cx+d)^2 + (cx+d)^2\frac{d}{dx}(ax+b) \\
&= (ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right) + (cx+d)^2\frac{d}{dx}(ax+b) \\
&= (ax+b)\left[\frac{d}{dx}\left(c^2x^2\right) + \frac{d}{dx}(2cdx) + \frac{d}{dx}d^2\right] + (cx+d)^2\left[\frac{d}{dx}ax + \frac{d}{dx}b\right] \\
&= (ax+b)\left(2c^2x+2cd\right) + \left(cx+d^2\right)a \\
&= 2c(ax+b)(cx+d) + a(cx+d)^2
\end{aligned}
$$

Question 5:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{ax+b}{cx+d}$

Answer :

Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$
$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$
$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$
$$=\frac{ad-bc}{(cx+d)^2}$$

Question 6:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Answer :

$$\text{Let } f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}, \text{ where } x\neq 0$$

By quotient rule,

$$\begin{aligned} f'(x)&=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, \ x\neq 0,\ 1 \\ &=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}, \ x\neq 0,\ 1 \\ &=\frac{x-1-x-1}{(x-1)^2}, \ x\neq 0,\ 1 \\ &=\frac{-2}{(x-1)^2}, \ x\neq 0,\ 1 \end{aligned}$$

Question 7:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{1}{ax^2+bx+c}$

Answer :

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$\begin{aligned} f'(x)&=\frac{(ax^2+bx+c)\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2} \\ &=\frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2} \\ &=\frac{-(2ax+b)}{(ax^2+bx+c)^2} \end{aligned}$$

Question 8:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{ax+b}{px^2+qx+r}$

Answer :

Let $f(x)=\dfrac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

Question 9:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{px^2+qx+r}{ax+b}$

Answer :

Let $f(x)=\dfrac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Answer :

$$\text{Let } f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}(x^{-4})-b\frac{d}{dx}(x^{-2})+\frac{d}{dx}(\cos x)$$

$$=a(-4x^{-5})-b(-2x^{-3})+(-\sin x)\qquad\left[\frac{d}{dx}(x^n)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Question 11:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $4\sqrt{x}-2$

Answer :

$$\text{Let } f(x) = 4\sqrt{x} - 2$$

$$f'(x) = \frac{d}{dx}\left(4\sqrt{x} - 2\right) = \frac{d}{dx}\left(4\sqrt{x}\right) - \frac{d}{dx}(2)$$

$$= 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 0 = 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= \left(2x^{-\frac{1}{2}}\right) = \frac{2}{\sqrt{x}}$$

Question 12:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax + b)^n$

Answer :

Let $f(x) = (ax+b)^n$. Accordingly, $f(x+h) = \{a(x+h)+b\}^n = (ax+ah+b)^n$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\lfloor 2}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

$$(\text{Using binomial theorem})$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{\lfloor 2(ax+b)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$

$$=(ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{\lfloor 2(ax+b)^2}+\ldots\right]$$

$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$=na\frac{(ax+b)^n}{(ax+b)}$$

$$=na(ax+b)^{n-1}$$

Question 13:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax+b)^n(cx+d)^m$

Answer :

Let $f(x)=(ax+b)^n(cx+d)^m$

By Leibnitz product rule,

$$f'(x)=(ax+b)^n \frac{d}{dx}(cx+d)^m+(cx+d)^m \frac{d}{dx}(ax+b)^n \qquad ...(1)$$

Now, let $f_1(x)=(cx+d)^m$

$$f_1(x+h)=(cx+ch+d)^m$$

$$\begin{aligned}
f_1'(x)&=\lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}\\
&=\lim_{h\to 0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}\\
&=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]\\
&=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+...\right)-1\right]\\
&=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+...(\text{Terms containing higher degrees of } h)\right]\\
&=(cx+d)^m \lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+...\right]\\
&=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]\\
&=\frac{mc(cx+d)^m}{(cx+d)}\\
&=mc(cx+d)^{m-1}
\end{aligned}$$

$$\frac{d}{dx}(cx+d)^m=mc(cx+d)^{m-1} \qquad ...(2)$$

Similarly, $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1} \qquad ...(3)$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned}
f'(x)&=(ax+b)^n\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^m\left\{na(ax+b)^{n-1}\right\}\\
&=(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]
\end{aligned}$$

Question 14:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): sin (*x* + *a*)

Answer :

Let $f(x)=\sin(x+a)$

$f(x+h)=\sin(x+h+a)$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]\\
&= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]\\
&= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} && \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]\\
&= \cos\left(\frac{2x+2a}{2}\right)\times 1 && \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\
&= \cos(x+a)
\end{aligned}$$

Question 15:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): cosec *x* cot *x*

Answer :

Let $f(x) = \operatorname{cosec} x \cot x$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x (\cot x)' + \cot x (\operatorname{cosec} x)' \qquad \ldots(1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x+h) = \cot(x+h)$

By first principle,

$$\begin{aligned}
f_1'(x) &= \lim_{h\to 0} \frac{f_1(x+h) - f_1(x)}{h} \\
&= \lim_{h\to 0} \frac{\cot(x+h) - \cot x}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)}\right] \\
&= \frac{1}{\sin x} . \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right] \\
&= \frac{-1}{\sin x} . \left(\lim_{h\to 0} \frac{\sin h}{h}\right)\left(\lim_{h\to 0} \frac{1}{\sin(x+h)}\right) \\
&= \frac{-1}{\sin x} . 1 . \left(\frac{1}{\sin(x+0)}\right) \\
&= \frac{-1}{\sin^2 x} \\
&= -\operatorname{cosec}^2 x
\end{aligned}$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \qquad \ldots(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$f_2'(x)=\lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}.\lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{-1}{\sin x}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=\frac{-1}{\sin x}.1.\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=\frac{-1}{\sin x}.\frac{\cos x}{\sin x}$$

$$=-\operatorname{cosec} x.\cot x$$

$$\therefore(\operatorname{cosec} x)'=-\operatorname{cosec} x.\cot x \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$f'(x)=\operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right)+\cot x\left(-\operatorname{cosec} x\cot x\right)$$

$$=-\operatorname{cosec}^3 x-\cot^2 x\ \operatorname{cosec} x$$

Question 16:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\cos x}{1+\sin x}$

Answer :

Let $f(x)=\frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)}$$

Question 17:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin x+\cos x}{\sin x-\cos x}$

Answer :

Let $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Question 18:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{\sec x-1}{\sec x+1}$

Answer :

Let $f(x)=\dfrac{\sec x-1}{\sec x+1}$

$$f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}=\frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$=\frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$=\frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

Question 19:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\sin^n x$

Answer :

Let $y=\sin^n x$.

Accordingly, for $n=1$, $y=\sin x$.

$$\therefore \frac{dy}{dx}=\cos x,\ \text{i.e.,}\ \frac{d}{dx}\sin x=\cos x$$

For $n=2$, $y=\sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x (\sin x)' \qquad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2 \sin x \cos x \qquad \ldots(1)$$

For $n = 3$, $y = \sin^3 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$$

$$= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' \qquad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin^2 x + \sin x (2 \sin x \cos x) \qquad \text{[Using (1)]}$$

$$= \cos x \sin^2 x + 2 \sin^2 x \cos x$$

$$= 3 \sin^2 x \cos x$$

We assert that $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

i.e., $\frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \qquad \ldots(2)$

Consider

$$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^k x)$$

$$= (\sin x)' \sin^k x + \sin x (\sin^k x)' \qquad \text{[By Leibnitz product rule]}$$

$$= \cos x \sin^k x + \sin x \left(k \sin^{(k-1)} x \cos x\right) \qquad \text{[Using (2)]}$$

$$= \cos x \sin^k x + k \sin^k x \cos x$$

$$= (k+1) \sin^k x \cos x$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Question 20:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a+b\sin x}{c+d\cos x}$

Answer :

Let $f(x)=\frac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$
$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$
$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$
$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$
$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

Question 21:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin(x+a)}{\cos x}$

Answer :

Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)(-\sin x)}{\cos^2 x} \qquad \ldots \text{(i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=\left(\cos\frac{2x+2a}{2}\right)\times 1 \qquad \left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=\cos(x+a) \qquad \ldots \text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$=\frac{\cos(x+a-x)}{\cos^2 x}$$

$$=\frac{\cos a}{\cos^2 x}$$

Question 22:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): x^4 (5 sin x – 3 cos x)

Answer :

Let $f(x) = x^4(5\sin x - 3\cos x)$

By product rule,

$$\begin{aligned} f'(x) &= x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4) \\ &= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4) \\ &= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3) \\ &= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right] \end{aligned}$$

Question 23:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (x^2 + 1) cos x

Answer :

Let $f(x) = (x^2+1)\cos x$

By product rule,

$$\begin{aligned} f'(x) &= (x^2+1)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2+1) \\ &= (x^2+1)(-\sin x) + \cos x(2x) \\ &= -x^2\sin x - \sin x + 2x\cos x \end{aligned}$$

Question 24:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax^2 + \sin x)(p + q\cos x)$

Answer :

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By product rule,

$$\begin{aligned} f'(x) &= (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x) \\ &= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x) \\ &= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x) \end{aligned}$$

Question 25:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(x + \cos x)(x - \tan x)$

Answer :

Let $f(x) = (x + \cos x)(x - \tan x)$

By product rule,

$$\begin{aligned} f'(x) &= (x + \cos x)\frac{d}{dx}(x - \tan x) + (x - \tan x)\frac{d}{dx}(x + \cos x) \\ &= (x + \cos x)\left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right] + (x - \tan x)(1 - \sin x) \\ &= (x + \cos x)\left[1 - \frac{d}{dx}\tan x\right] + (x - \tan x)(1 - \sin x) \qquad \ldots \text{(i)} \end{aligned}$$

Let $g(x) = \tan x$. Accordingly, $g(x + h) = \tan(x + h)$

By first principle,

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right) \\
&= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \qquad \text{... (ii)}
\end{aligned}
$$

Therefore, from (i) and (ii), we obtain

$$
\begin{aligned}
f'(x) &= (x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x) \\
&= (x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x) \\
&= -\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)
\end{aligned}
$$

Question 26:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Answer :

Let $f(x)=\dfrac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer :

Let $f(x)=\dfrac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x) = \cos\frac{\pi}{4}.\left[\frac{\sin x \frac{d}{dx}\left(x^2\right) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$= \cos\frac{\pi}{4}.\left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x}\right]$$

$$= \frac{x\cos\frac{\pi}{4}\left[2\sin x - x\cos x\right]}{\sin^2 x}$$

Question 28:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{1+\tan x}$

Answer :

Let $f(x) = \frac{x}{1+\tan x}$

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x) = \frac{(1+\tan x) - x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \text{... (i)}$$

Let $g(x) = 1+\tan x$. Accordingly, $g(x+h) = 1+\tan(x+h)$.

By first principle,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{1 + \tan(x+h) - 1 - \tan x}{h} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)\cos x} \right]$$

$$= \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \to 0} \frac{1}{\cos(x+h)\cos x} \right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1 + \tan x) = \sec^2 x \qquad \text{... (ii)}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

Question 29:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (x + sec x) (x – tan x)

Answer :

Let $f(x) = (x + \sec x)(x - \tan x)$

By product rule,

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$

$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$=\lim_{h\to0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$=\lim_{h\to0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to0}\frac{\sin h}{h}\right).\left(\lim_{h\to0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}\tan x=\sec^2 x \qquad \text{... (ii)}$$

$$f_2'(x)=\lim_{h\to 0}\left(\frac{f_2(x+h)-f_2(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]$$

$$=\sec x.\frac{\left\{\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h\to 0}\cos(x+h)}$$

$$=\sec x.\frac{\sin x.1}{\cos x}$$

$$\Rightarrow\frac{d}{dx}\sec x=\sec x\tan x \qquad .. \text{(iii)}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{\sin^n x}$

Answer :

Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$\begin{aligned} f'(x) &= \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x} \\ &= \frac{\sin^n x.1 - x\left(n \sin^{n-1} x \cos x\right)}{\sin^{2n} x} \\ &= \frac{\sin^{n-1} x (\sin x - nx \cos x)}{\sin^{2n} x} \\ &= \frac{\sin x - nx \cos x}{\sin^{n+1} x} \end{aligned}$$