Chapter 10 Straight Lines

Miscellaneous

Question 1:

Find the values of k for which the line $(k-3)x-(4-k^2)y+k^2-7k+6=0$ is

(a) Parallel to the x-axis,

(b) Parallel to the y-axis,

(c) Passing through the origin.

Answer :

The given equation of line is

$(k-3)x-(4-k^2)y+k^2-7k+6=0$ … (1)

(a) If the given line is parallel to the x-axis, then

Slope of the given line = Slope of the x-axis

The given line can be written as

$(4-k^2)y=(k-3)x+k^2-7k+6=0$

$y=\frac{(k-3)}{(4-k^2)}x+\frac{k^2-7k+6}{(4-k^2)}$, which is of the form $y = mx + c$.

$\therefore$Slope of the given line = $\frac{(k-3)}{(4-k^2)}$

Slope of the x-axis = 0

$$\therefore \frac{(k-3)}{(4-k^2)}=0$$
$$\Rightarrow k-3=0$$
$$\Rightarrow k=3$$

Thus, if the given line is parallel to the x-axis, then the value of k is 3.

(b) If the given line is parallel to the y-axis, it is vertical. Hence, its slope will be undefined.

The slope of the given line is $\frac{(k-3)}{(4-k^2)}$.

Now, $\frac{(k-3)}{(4-k^2)}$ is undefined at $k^2 = 4$

$k^2 = 4$

$\Rightarrow k = \pm 2$

Thus, if the given line is parallel to the y-axis, then the value of k is ±2.

(c) If the given line is passing through the origin, then point (0, 0) satisfies the

given equation of line.

$$
\begin{aligned}
&(k-3)(0)-(4-k^2)(0)+k^2-7k+6=0\\
&k^2-7k+6=0\\
&k^2-6k-k+6=0\\
&(k-6)(k-1)=0\\
&k=1 \text{ or } 6
\end{aligned}
$$

Thus, if the given line is passing through the origin, then the value of k is either 1 or 6.

Question 2:

Find the values of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

Answer :

The equation of the given line is $\sqrt{3}x + y + 2 = 0$.

This equation can be reduced as

$$\sqrt{3}x + y + 2 = 0$$
$$\Rightarrow -\sqrt{3}x - y = 2$$

On dividing both sides by $\sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$, we obtain

$$-\frac{\sqrt{3}}{2}x - \frac{1}{2}y = \frac{2}{2}$$
$$\Rightarrow \left(-\frac{\sqrt{3}}{2}\right)x + \left(-\frac{1}{2}\right)y = 1 \qquad ...(1)$$

On comparing equation (1) to $x\cos\theta + y\sin\theta = p$, we obtain

$$\cos\theta = -\frac{\sqrt{3}}{2},\ \sin\theta = -\frac{1}{2}, \text{ and } p = 1$$

Since the values of sin *θ* and cos *θ* are negative, $\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

Thus, the respective values of *θ*and *p* are $\frac{7\pi}{6}$ and 1

Question 3:

Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and –6, respectively.

Answer :

Let the intercepts cut by the given lines on the axes be *a* and *b*.

It is given that

$a + b = 1$ … (1)

$ab = –6$ … (2)

On solving equations (1) and (2), we obtain

$a = 3$ and $b = –2$ or $a = –2$ and $b = 3$

It is known that the equation of the line whose intercepts on the axes are *a* and *b* is

$$\frac{x}{a}+\frac{y}{b}=1 \text{ or } bx+ay-ab=0$$

Case I: $a = 3$ and $b = –2$

In this case, the equation of the line is $–2x + 3y + 6 = 0$, i.e., $2x – 3y = 6$.

Case II: $a = –2$ and $b = 3$

In this case, the equation of the line is $3x – 2y + 6 = 0$, i.e., $–3x + 2y = 6$.

Thus, the required equation of the lines are $2x – 3y = 6$ and $–3x + 2y = 6$.

Question 4:

What are the points on the *y*-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.

Answer :

Let $(0, b)$ be the point on the *y*-axis whose distance from line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.

The given line can be written as $4x + 3y – 12 = 0$ … (1)

On comparing equation (1) to the general equation of line $Ax + By + C = 0$, we obtain $A = 4$, $B = 3$, and $C = –12$.

It is known that the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$.

Therefore, if $(0, b)$ is the point on the *y*-axis whose distance from line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units, then:

$$4=\frac{|4(0)+3(b)-12|}{\sqrt{4^2+3^2}}$$
$$\Rightarrow 4=\frac{|3b-12|}{5}$$
$$\Rightarrow 20=|3b-12|$$
$$\Rightarrow 20=\pm(3b-12)$$
$$\Rightarrow 20=(3b-12) \text{ or } 20=-(3b-12)$$
$$\Rightarrow 3b=20+12 \text{ or } 3b=-20+12$$
$$\Rightarrow b=\frac{32}{3} \text{ or } b=-\frac{8}{3}$$

Thus, the required points are $\left(0,\frac{32}{3}\right)$ and $\left(0,-\frac{8}{3}\right)$.

Question 5:

Find the perpendicular distance from the origin to the line joining the points $(\cos\theta,\sin\theta)$ and $(\cos\phi,\sin\phi)$.

Answer :

The equation of the line joining the points $(\cos\theta,\sin\theta)$ and $(\cos\phi,\sin\phi)$ is given by

$$y-\sin\theta=\frac{\sin\phi-\sin\theta}{\cos\phi-\cos\theta}(x-\cos\theta)$$
$$y(\cos\phi-\cos\theta)-\sin\theta(\cos\phi-\cos\theta)=x(\sin\phi-\sin\theta)-\cos\theta(\sin\phi-\sin\theta)$$
$$x(\sin\theta-\sin\phi)+y(\cos\phi-\cos\theta)+\cos\theta\sin\phi-\cos\theta\sin\theta-\sin\theta\cos\phi+\sin\theta\cos\theta=0$$
$$x(\sin\theta-\sin\phi)+y(\cos\phi-\cos\theta)+\sin(\phi-\theta)=0$$
$$Ax+By+C=0,\text{ where } A=\sin\theta-\sin\phi,\ B=\cos\phi-\cos\theta,\text{ and } C=\sin(\phi-\theta)$$

It is known that the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$.

Therefore, the perpendicular distance (d) of the given line from point $(x_1, y_1) = (0, 0)$ is

$$d=\frac{\left|(\sin\theta-\sin\phi)(0)+(\cos\phi-\cos\theta)(0)+\sin(\phi-\theta)\right|}{\sqrt{(\sin\theta-\sin\phi)^2+(\cos\phi-\cos\theta)^2}}$$

$$=\frac{\left|\sin(\phi-\theta)\right|}{\sqrt{\sin^2\theta+\sin^2\phi-2\sin\theta\sin\phi+\cos^2\phi+\cos^2\theta-2\cos\phi\cos\theta}}$$

$$=\frac{\left|\sin(\phi-\theta)\right|}{\sqrt{(\sin^2\theta+\cos^2\theta)+(\sin^2\phi+\cos^2\phi)-2(\sin\theta\sin\phi+\cos\theta\cos\phi)}}$$

$$=\frac{\left|\sin(\phi-\theta)\right|}{\sqrt{1+1-2(\cos(\phi-\theta))}}$$

$$=\frac{\left|\sin(\phi-\theta)\right|}{\sqrt{2(1-\cos(\phi-\theta))}}$$

$$=\frac{\left|\sin(\phi-\theta)\right|}{\sqrt{2\left(2\sin^2\left(\frac{\phi-\theta}{2}\right)\right)}}$$

$$=\frac{\left|\sin(\phi-\theta)\right|}{\left|2\sin\left(\frac{\phi-\theta}{2}\right)\right|}$$

Question 6:

Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x – 7y + 5 = 0$ and $3x + y = 0$.

Answer :

The equation of any line parallel to the y-axis is of the form

$x = a$ … (1)

The two given lines are

$x – 7y + 5 = 0$ … (2)

$3x + y = 0$ … (3)

On solving equations (2) and (3), we obtain $x = -\frac{5}{22}$ and $y = \frac{15}{22}$.

Therefore, $\left(-\frac{5}{22}, \frac{15}{22}\right)$ is the point of intersection of lines (2) and (3).

Since line $x = a$ passes through point $\left(-\frac{5}{22}, \frac{15}{22}\right)$, $a = -\frac{5}{22}$.

Thus, the required equation of the line is $x = -\frac{5}{22}$.

Question 7:

Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point, where it meets the y-axis.

Answer :

The equation of the given line is $\frac{x}{4} + \frac{y}{6} = 1$.

This equation can also be written as $3x + 2y - 12 = 0$

$y = \frac{-3}{2}x + 6$, which is of the form $y = mx + c$

∴Slope of the given line $= -\frac{3}{2}$

∴Slope of line perpendicular to the given line $= -\frac{1}{\left(-\frac{3}{2}\right)} = \frac{2}{3}$

Let the given line intersect the y-axis at $(0, y)$.

On substituting x with 0 in the equation of the given line, we obtain $\frac{y}{6} = 1 \Rightarrow y = 6$

∴The given line intersects the y-axis at (0, 6).

The equation of the line that has a slope of $\frac{2}{3}$ and passes through point (0, 6) is

$$(y-6)=\frac{2}{3}(x-0)$$
$$3y-18=2x$$
$$2x-3y+18=0$$

Thus, the required equation of the line is $2x-3y+18=0$.

Question 8:

Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.

Answer :

The equations of the given lines are

$y - x = 0$ … (1)

$x + y = 0$ … (2)

$x - k = 0$ … (3)

The point of intersection of lines (1) and (2) is given by

$x = 0$ and $y = 0$

The point of intersection of lines (2) and (3) is given by

$x = k$ and $y = -k$

The point of intersection of lines (3) and (1) is given by

$x = k$ and $y = k$

Thus, the vertices of the triangle formed by the three given lines are (0, 0), (k, $-k$), and (k, k).

We know that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|.$$

Therefore, area of the triangle formed by the three given lines

$$=\frac{1}{2}\left|0(-k-k)+k(k-0)+k(0+k)\right| \text{ square units}$$
$$=\frac{1}{2}\left|k^2+k^2\right| \text{ square units}$$
$$=\frac{1}{2}\left|2k^2\right| \text{ square units}$$
$$=k^2 \text{ square units}$$

Question 9:

Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

Answer :

The equations of the given lines are

$3x + y - 2 = 0$ … (1)

$px + 2y - 3 = 0$ … (2)

$2x - y - 3 = 0$ … (3)

On solving equations (1) and (3), we obtain

$x = 1$ and $y = -1$

Since these three lines may intersect at one point, the point of intersection of lines (1) and (3) will also satisfy line (2).

$p(1) + 2(-1) - 3 = 0$

$p - 2 - 3 = 0$

$p = 5$

Thus, the required value of p is 5.

Question 10:

If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.

Answer :

The equations of the given lines are

$y = m_1x + c_1$ … (1)

$y = m_2x + c_2$ … (2)

$y = m_3x + c_3$ … (3)

On subtracting equation (1) from (2), we obtain

$$0 = (m_2 - m_1)x + (c_2 - c_1)$$
$$\Rightarrow (m_1 - m_2)x = c_2 - c_1$$
$$\Rightarrow x = \frac{c_2 - c_1}{m_1 - m_2}$$

On substituting this value of x in (1), we obtain

$$y = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1$$
$$y = \frac{m_1c_2 - m_1c_1}{m_1 - m_2} + c_1$$
$$y = \frac{m_1c_2 - m_1c_1 + m_1c_1 - m_2c_1}{m_1 - m_2}$$
$$y = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

$\therefore \left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$ is the point of intersection of lines (1) and (2).

It is given that lines (1), (2), and (3) are concurrent. Hence, the point of intersection of lines (1) and (2) will also satisfy equation (3).

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = m_3\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_3$$

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = \frac{m_3c_2 - m_3c_1 + c_3m_1 - c_3m_2}{m_1 - m_2}$$

$$m_1c_2 - m_2c_1 - m_3c_2 + m_3c_1 - c_3m_1 + c_3m_2 = 0$$

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$

Hence, $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$

Question 11:

Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.

Answer :

Let the slope of the required line be m_1.

The given line can be written as $y = \frac{1}{2}x - \frac{3}{2}$, which is of the form $y = mx + c$

∴Slope of the given line = $m_2 = \frac{1}{2}$

It is given that the angle between the required line and line $x - 2y = 3$ is 45°.

We know that if θisthe acute angle between lines l_1 and l_2 with slopes m_1 and m_2 respectively, then $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$.

$$\therefore \tan 45° = \frac{|m_1 - m_2|}{1 + m_1 m_2}$$

$$\Rightarrow 1 = \left|\frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}\right|$$

$$\Rightarrow 1 = \left|\frac{\left(\frac{1-2m_1}{2}\right)}{\frac{2+m_1}{2}}\right|$$

$$\Rightarrow 1 = \left|\frac{1-2m_1}{2+m_1}\right|$$

$$\Rightarrow 1 = \pm\left(\frac{1-2m_1}{2+m_1}\right)$$

$$\Rightarrow 1 = \frac{1-2m_1}{2+m_1} \text{ or } 1 = -\left(\frac{1-2m_1}{2+m_1}\right)$$

$$\Rightarrow 2 + m_1 = 1 - 2m_1 \text{ or } 2 + m_1 = -1 + 2m_1$$

$$\Rightarrow m_1 = -\frac{1}{3} \text{ or } m_1 = 3$$

Case I: $m_1 = 3$

The equation of the line passing through (3, 2) and having a slope of 3 is:

$y - 2 = 3\,(x - 3)$

$y - 2 = 3x - 9$

$3x - y = 7$

Case II: $m_1 = -\frac{1}{3}$

The equation of the line passing through (3, 2) and having a slope of $-\frac{1}{3}$ is:

$$y-2=-\frac{1}{3}(x-3)$$
$$3y-6=-x+3$$
$$x+3y=9$$

Thus, the equations of the lines are $3x - y = 7$ and $x + 3y = 9$.

Question 12:

Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

Answer :

Let the equation of the line having equal intercepts on the axes be

$$\frac{x}{a}+\frac{y}{a}=1$$
$$\text{Or } x+y=a \qquad \ldots(1)$$

On solving equations $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$, we obtain $x=\frac{1}{13}$ and $y=\frac{5}{13}$.

$\therefore \left(\frac{1}{13},\frac{5}{13}\right)$ is the point of intersection of the two given lines.

Since equation (1) passes through point $\left(\frac{1}{13},\frac{5}{13}\right)$,

$$\frac{1}{13}+\frac{5}{13}=a$$
$$\Rightarrow a=\frac{6}{13}$$

$\therefore$ Equation (1) becomes $x+y=\frac{6}{13}$, i.e., $13x+13y=6$

Thus, the required equation of the line is $13x+13y=6$.

Question 13:

Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$.

Answer :

Let the equation of the line passing through the origin be $y = m_1 x$.

If this line makes an angle of θ with line $y = mx + c$, then angle θ is given by

$$\therefore \tan\theta = \left|\frac{m_1 - m}{1 + m_1 m}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right|$$

$$\Rightarrow \tan\theta = \pm\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

$$\Rightarrow \tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m} \text{ or } \tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

Case I: $\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}$

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}$$

$$\Rightarrow \tan\theta + \frac{y}{x}m\tan\theta = \frac{y}{x} - m$$

$$\Rightarrow m + \tan\theta = \frac{y}{x}(1 - m\tan\theta)$$

$$\Rightarrow \frac{y}{x} = \frac{m + \tan\theta}{1 - m\tan\theta}$$

Case II: $\tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$

$$\tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

$$\Rightarrow \tan\theta + \frac{y}{x}m\tan\theta = -\frac{y}{x} + m$$

$$\Rightarrow \frac{y}{x}(1 + m\tan\theta) = m - \tan\theta$$

$$\Rightarrow \frac{y}{x} = \frac{m - \tan\theta}{1 + m\tan\theta}$$

Therefore, the required line is given by $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$.

Question 14:

In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line

$x + y = 4$?

Answer :

The equation of the line joining the points (–1, 1) and (5, 7) is given by

$$y-1=\frac{7-1}{5+1}(x+1)$$
$$y-1=\frac{6}{6}(x+1)$$
$$x-y+2=0 \qquad ...(1)$$

The equation of the given line is

$x+y-4=0$ … (2)

The point of intersection of lines (1) and (2) is given by

$x = 1$ and $y = 3$

Let point (1, 3) divide the line segment joining (–1, 1) and (5, 7) in the ratio 1:k. Accordingly, by section formula,

$$(1,3)=\left(\frac{k(-1)+1(5)}{1+k},\frac{k(1)+1(7)}{1+k}\right)$$
$$\Rightarrow (1,3)=\left(\frac{-k+5}{1+k},\frac{k+7}{1+k}\right)$$
$$\Rightarrow \frac{-k+5}{1+k}=1,\frac{k+7}{1+k}=3$$
$$\therefore \frac{-k+5}{1+k}=1$$
$$\Rightarrow -k+5=1+k$$
$$\Rightarrow 2k=4$$
$$\Rightarrow k=2$$

Thus, the line joining the points (–1, 1) and (5, 7) is divided by line

$x + y = 4$ in the ratio 1:2.

Question 15:

Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.

Answer :

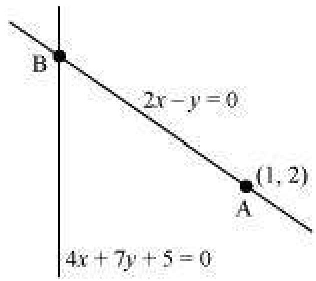
The given lines are

$2x - y = 0$ … (1)

$4x + 7y + 5 = 0$ … (2)

A (1, 2) is a point on line (1).

Let B be the point of intersection of lines (1) and (2).



On solving equations (1) and (2), we obtain $x = \frac{-5}{18}$ and $y = \frac{-5}{9}$.

∴Coordinates of point B are $\left(\frac{-5}{18}, \frac{-5}{9}\right)$.

By using distance formula, the distance between points A and B can be obtained as

$$AB = \sqrt{\left(1+\frac{5}{18}\right)^2 + \left(2+\frac{5}{9}\right)^2} \text{ units}$$

$$= \sqrt{\left(\frac{23}{18}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

$$= \sqrt{\left(\frac{23}{2\times 9}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

$$= \sqrt{\left(\frac{23}{9}\right)^2 \left(\frac{1}{2}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

$$= \sqrt{\left(\frac{23}{9}\right)^2 \left(\frac{1}{4}+1\right)} \text{ units}$$

$$= \frac{23}{9}\sqrt{\frac{5}{4}} \text{ units}$$

$$= \frac{23}{9} \times \frac{\sqrt{5}}{2} \text{ units}$$

$$= \frac{23\sqrt{5}}{18} \text{ units}$$

Thus, the required distance is $\frac{23\sqrt{5}}{18}$ units.

Question 16:

Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

Answer :

Let $y = mx + c$ be the line through point (–1, 2).

Accordingly, $2 = m(-1) + c$.

$\Rightarrow 2 = -m + c$

$\Rightarrow c = m + 2$

$\therefore y = mx + m + 2$ … (1)

The given line is

$x + y = 4$ … (2)

On solving equations (1) and (2), we obtain

$$x = \frac{2-m}{m+1} \text{ and } y = \frac{5m+2}{m+1}$$

$\therefore \left(\frac{2-m}{m+1}, \frac{5m+2}{m+1}\right)$ is the point of intersection of lines (1) and (2).

Since this point is at a distance of 3 units from point (– 1, 2), according to distance formula,

$$\sqrt{\left(\frac{2-m}{m+1}+1\right)^2 + \left(\frac{5m+2}{m+1}-2\right)^2} = 3$$
$$\Rightarrow \left(\frac{2-m+m+1}{m+1}\right)^2 + \left(\frac{5m+2-2m-2}{m+1}\right)^2 = 3^2$$
$$\Rightarrow \frac{9}{(m+1)^2} + \frac{9m^2}{(m+1)^2} = 9$$
$$\Rightarrow \frac{1+m^2}{(m+1)^2} = 1$$
$$\Rightarrow 1+m^2 = m^2+1+2m$$
$$\Rightarrow 2m = 0$$
$$\Rightarrow m = 0$$

Thus, the slope of the required line must be zero i.e., the line must be parallel to the *x*-axis.

Question 18:

Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

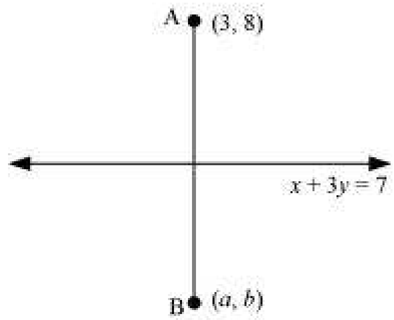
Answer :

The equation of the given line is

$x + 3y = 7$ … (1)

Let point B (a, b) be the image of point A (3, 8).

Accordingly, line (1) is the perpendicular bisector of AB.



Slope of AB $= \frac{b-8}{a-3}$, while the slope of line $(1) = -\frac{1}{3}$

Since line (1) is perpendicular to AB,

$$\left(\frac{b-8}{a-3}\right)\times\left(-\frac{1}{3}\right) = -1$$
$$\Rightarrow \frac{b-8}{3a-9} = 1$$
$$\Rightarrow b-8 = 3a-9$$
$$\Rightarrow 3a-b = 1 \qquad ...(2)$$

Mid-point of AB $= \left(\frac{a+3}{2}, \frac{b+8}{2}\right)$

The mid-point of line segment AB will also satisfy line (1).

Hence, from equation (1), we have

$$\left(\frac{a+3}{2}\right)+3\left(\frac{b+8}{2}\right) = 7$$
$$\Rightarrow a+3+3b+24 = 14$$
$$\Rightarrow a+3b = -13 \qquad ...(3)$$

On solving equations (2) and (3), we obtain $a = -1$ and $b = -4$.

Thus, the image of the given point with respect to the given line is (–1, –4).

Question 19:

If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

Answer :

The equations of the given lines are

$y = 3x + 1$ … (1)

$2y = x + 3$ … (2)

$y = mx + 4$ … (3)

Slope of line (1), $m_1 = 3$

Slope of line (2), $m_2 = \frac{1}{2}$

Slope of line (3), $m_3 = m$

It is given that lines (1) and (2) are equally inclined to line (3). This means that

the angle between lines (1) and (3) equals the angle between lines (2) and (3).

$$\therefore \left|\frac{m_1 - m_3}{1 + m_1 m_3}\right| = \left|\frac{m_2 - m_3}{1 + m_2 m_3}\right|$$

$$\Rightarrow \left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{\frac{1}{2} - m}{1 + \frac{1}{2}m}\right|$$

$$\Rightarrow \left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{1 - 2m}{m + 2}\right|$$

$$\Rightarrow \frac{3 - m}{1 + 3m} = \pm\left(\frac{1 - 2m}{m + 2}\right)$$

$$\Rightarrow \frac{3-m}{1+3m} = \frac{1-2m}{m+2} \text{ or } \frac{3-m}{1+3m} = -\left(\frac{1-2m}{m+2}\right)$$

If $\frac{3-m}{1+3m} = \frac{1-2m}{m+2}$, then

$$(3-m)(m+2) = (1-2m)(1+3m)$$
$$\Rightarrow -m^2 + m + 6 = 1 + m - 6m^2$$
$$\Rightarrow 5m^2 + 5 = 0$$
$$\Rightarrow (m^2 + 1) = 0$$
$$\Rightarrow m = \sqrt{-1}, \text{ which is not real}$$

Hence, this case is not posible.

If $\frac{3-m}{1+3m} = -\left(\frac{1-2m}{m+2}\right)$, then

$$\Rightarrow (3-m)(m+2) = -(1-2m)(1+3m)$$
$$\Rightarrow -m^2 + m + 6 = -(1 + m - 6m^2)$$
$$\Rightarrow 7m^2 - 2m - 7 = 0$$
$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4(7)(-7)}}{2(7)}$$
$$\Rightarrow m = \frac{2 \pm 2\sqrt{1+49}}{14}$$
$$\Rightarrow m = \frac{1 \pm 5\sqrt{2}}{7}$$

Thus, the required value of m is $\frac{1 \pm 5\sqrt{2}}{7}$.

Question 20:

If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

Answer :

The equations of the given lines are

$x + y - 5 = 0$ … (1)

$3x - 2y + 7 = 0$ … (2)

The perpendicular distances of P (x, y) from lines (1) and (2) are respectively given by

$$d_1 = \frac{|x + y - 5|}{\sqrt{(1)^2 + (1)^2}} \text{ and } d_2 = \frac{|3x - 2y + 7|}{\sqrt{(3)^2 + (-2)^2}}$$

$$\text{i.e., } d_1 = \frac{|x + y - 5|}{\sqrt{2}} \text{ and } d_2 = \frac{|3x - 2y + 7|}{\sqrt{13}}$$

It is given that $d_1 + d_2 = 10$.

$$\therefore \frac{|x + y - 5|}{\sqrt{2}} + \frac{|3x - 2y + 7|}{\sqrt{13}} = 10$$
$$\Rightarrow \sqrt{13}|x + y - 5| + \sqrt{2}|3x - 2y + 7| - 10\sqrt{26} = 0$$
$$\Rightarrow \sqrt{13}(x + y - 5) + \sqrt{2}(3x - 2y + 7) - 10\sqrt{26} = 0$$
$$[\text{Assuming } (x + y - 5) \text{ and } (3x - 2y + 7) \text{ are positive}]$$
$$\Rightarrow \sqrt{13}x + \sqrt{13}y - 5\sqrt{13} + 3\sqrt{2}x - 2\sqrt{2}y + 7\sqrt{2} - 10\sqrt{26} = 0$$

$\Rightarrow x\left(\sqrt{13} + 3\sqrt{2}\right) + y\left(\sqrt{13} - 2\sqrt{2}\right) + \left(7\sqrt{2} - 5\sqrt{13} - 10\sqrt{26}\right) = 0$, which is the equation of a line.

Similarly, we can obtain the equation of line for any signs of $(x + y - 5)$ and $(3x - 2y + 7)$.

Thus, point P must move on a line.

Question 21:

Find equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

Answer :

The equations of the given lines are

$9x + 6y - 7 = 0$ … (1)

$3x + 2y + 6 = 0$ … (2)

Let P (*h*, *k*) be the arbitrary point that is equidistant from lines (1) and (2). The perpendicular distance of P (*h*, *k*) from line (1) is given by

$$d_1 = \frac{|9h+6k-7|}{(9)^2+(6)^2} = \frac{|9h+6k-7|}{\sqrt{117}} = \frac{|9h+6k-7|}{3\sqrt{13}}$$

The perpendicular distance of P (*h, k*) from line (2) is given by

$$d_2 = \frac{|3h+2k+6|}{\sqrt{(3)^2+(2)^2}} = \frac{|3h+2k+6|}{\sqrt{13}}$$

Since P (*h*, *k*) is equidistant from lines (1) and (2), $d_1 = d_2$

$$\therefore \frac{|9h+6k-7|}{3\sqrt{13}} = \frac{|3h+2k+6|}{\sqrt{13}}$$

$$\Rightarrow |9h+6k-7| = 3|3h+2k+6|$$

$$\Rightarrow |9h+6k-7| = \pm 3(3h+2k+6)$$

$$\Rightarrow 9h+6k-7 = 3(3h+2k+6) \text{ or } 9h+6k-7 = -3(3h+2k+6)$$

The case $9h+6k-7 = 3(3h+2k+6)$ is not possible as

$9h+6k-7 = 3(3h+2k+6) \Rightarrow -7 = 18$ (which is absurd)

$\therefore 9h+6k-7 = -3(3h+2k+6)$

$9h + 6k - 7 = -9h - 6k - 18$

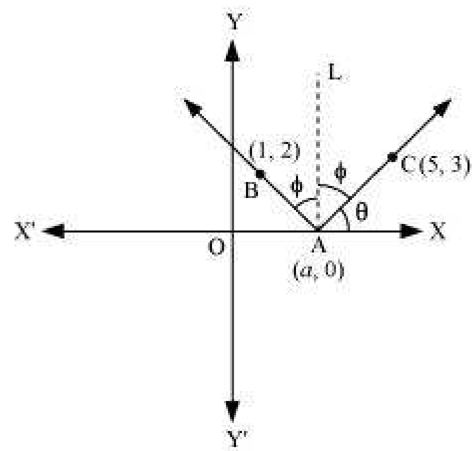
$\Rightarrow 18h + 12k + 11 = 0$

Thus, the required equation of the line is $18x + 12y + 11 = 0$.

Question 22:

A ray of light passing through the point (1, 2) reflects on the *x*-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

Answer :



Let the coordinates of point A be $(a, 0)$.

Draw a line (AL) perpendicular to the x-axis.

We know that angle of incidence is equal to angle of reflection. Hence, let

$\angle BAL = \angle CAL = \Phi$

Let $\angle CAX = \theta$

$\therefore \angle OAB = 180° - (\theta + 2\Phi) = 180° - [\theta + 2(90° - \theta)]$

$= 180° - \theta - 180° + 2\theta$

$= \theta$

$\therefore \angle BAX = 180° - \theta$

Now, slope of line AC $= \dfrac{3-0}{5-a}$

$$\Rightarrow \tan\theta = \frac{3}{5-a} \qquad ...(1)$$

Slope of line AB $= \dfrac{2-0}{1-a}$

$$\Rightarrow \tan(180° - \theta) = \frac{2}{1-a}$$

$$\Rightarrow -\tan\theta = \frac{2}{1-a}$$

$$\Rightarrow \tan\theta = \frac{2}{a-1} \qquad ...(2)$$

From equations (1) and (2), we obtain

$$\frac{3}{5-a}=\frac{2}{a-1}$$
$$\Rightarrow 3a-3=10-2a$$
$$\Rightarrow a=\frac{13}{5}$$

Thus, the coordinates of point A are $\left(\frac{13}{5},0\right)$.

Question 23:

Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2-b^2},0\right)$ and $\left(-\sqrt{a^2-b^2},0\right)$ to the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ is b^2.

Answer :

The equation of the given line is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$
$$\text{Or, } bx\cos\theta+ay\sin\theta-ab=0 \qquad \ldots(1)$$

Length of the perpendicular from point $\left(\sqrt{a^2-b^2},0\right)$ to line (1) is

$$p_1=\frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \qquad \ldots(2)$$

Length of the perpendicular from point $\left(-\sqrt{a^2-b^2},0\right)$ to line (2) is

$$p_2=\frac{\left|b\cos\theta\left(-\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\left|b\cos\theta\sqrt{a^2-b^2}+ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \qquad \ldots(3)$$

On multiplying equations (2) and (3), we obtain

$$p_1p_2 = \frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|\left|\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}\right)^2}$$

$$= \frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}-ab\right)\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}\right)^2-(ab)^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|b^2\cos^2\theta\left(a^2-b^2\right)-a^2b^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= \frac{\left|a^2b^2\cos^2\theta-b^4\cos^2\theta-a^2b^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\sin^2\theta-a^2\cos^2\theta\right|}{b^2\cos^2\theta+a^2\sin^2\theta} \qquad \left[\sin^2\theta+\cos^2\theta=1\right]$$

$$= \frac{b^2\left|-\left(b^2\cos^2\theta+a^2\sin^2\theta\right)\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{b^2\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

$$= b^2$$

Hence, proved.

Question 24:

A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Answer :

The equations of the given lines are

$2x - 3y + 4 = 0$ … (1)

$3x + 4y - 5 = 0$ … (2)

$6x - 7y + 8 = 0$ … (3)

The person is standing at the junction of the paths represented by lines (1) and (2).

On solving equations (1) and (2), we obtain $x = -\frac{1}{17}$ and $y = \frac{22}{17}$.

Thus, the person is standing at point $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

The person can reach path (3) in the least time if he walks along the perpendicular line to (3) from point $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Slope of the line (3) $= \frac{6}{7}$

∴Slope of the line perpendicular to line (3) $= -\frac{1}{\left(\frac{6}{7}\right)} = -\frac{7}{6}$

The equation of the line passing through $\left(-\frac{1}{17}, \frac{22}{17}\right)$ and having a slope of $-\frac{7}{6}$ is given by

$$\left(y - \frac{22}{17}\right) = -\frac{7}{6}\left(x + \frac{1}{17}\right)$$
$$6(17y - 22) = -7(17x + 1)$$
$$102y - 132 = -119x - 7$$
$$119x + 102y = 125$$

Hence, the path that the person should follow is $119x + 102y = 125$.