Chapter-3 Trigonometric Functions

Miscellaneous

Question 1:

Prove that: $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Answer :

L.H.S.

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\left(\frac{\frac{3\pi}{13}+\frac{5\pi}{13}}{2}\right)\cos\left(\frac{\frac{3\pi}{13}-\frac{5\pi}{13}}{2}\right) \quad \left[\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\left(\frac{-\pi}{13}\right)$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$$

$$=2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right]$$

$$=2\cos\frac{\pi}{13}\left[2\cos\left(\frac{\frac{9\pi}{13}+\frac{4\pi}{13}}{2}\right)\cos\left(\frac{\frac{9\pi}{13}-\frac{4\pi}{13}}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right]$$

$$=2\cos\frac{\pi}{13}\times 2\times 0\times\cos\frac{5\pi}{26}$$

= 0 = R.H.S

Question 2:

Prove that: (sin $3x$ + sin x) sin x + (cos $3x$ – cos x) cos x = 0

Answer :

L.H.S.

= (sin 3*x* + sin *x*) sin *x* + (cos 3*x* – cos *x*) cos *x*

$$= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x$$
$$= \cos 3x \cos x + \sin 3x \sin x - \left(\cos^2 x - \sin^2 x\right)$$
$$= \cos\left(3x - x\right) - \cos 2x \qquad \left[\cos\left(A - B\right) = \cos A \cos B + \sin A \sin B\right]$$
$$= \cos 2x - \cos 2x$$
$$= 0$$

= RH.S.

Question 3:

Prove that: $\left(\cos x + \cos y\right)^2 + \left(\sin x - \sin y\right)^2 = 4\cos^2 \frac{x+y}{2}$

Answer :

L.H.S. = $\left(\cos x + \cos y\right)^2 + \left(\sin x - \sin y\right)^2$

$$= \cos^2 x + \cos^2 y + 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y$$
$$= \left(\cos^2 x + \sin^2 x\right) + \left(\cos^2 y + \sin^2 y\right) + 2\left(\cos x \cos y - \sin x \sin y\right)$$
$$= 1 + 1 + 2\cos\left(x + y\right) \qquad \left[\cos\left(A + B\right) = \left(\cos A \cos B - \sin A \sin B\right)\right]$$
$$= 2 + 2\cos\left(x + y\right)$$
$$= 2\left[1 + \cos\left(x + y\right)\right]$$
$$= 2\left[1 + 2\cos^2\left(\frac{x+y}{2}\right) - 1\right] \qquad \left[\cos 2A = 2\cos^2 A - 1\right]$$
$$= 4\cos^2\left(\frac{x+y}{2}\right) = \text{R.H.S.}$$

Question 4:

Prove that: $(\cos x-\cos y)^2+(\sin x-\sin y)^2=4\sin^2\frac{x-y}{2}$

Answer :

L.H.S. = $(\cos x-\cos y)^2+(\sin x-\sin y)^2$

$$=\cos^2 x+\cos^2 y-2\cos x\cos y+\sin^2 x+\sin^2 y-2\sin x\sin y$$
$$=(\cos^2 x+\sin^2 x)+(\cos^2 y+\sin^2 y)-2[\cos x\cos y+\sin x\sin y]$$
$$=1+1-2[\cos(x-y)] \qquad [\cos(A-B)=\cos A\cos B+\sin A\sin B]$$
$$=2[1-\cos(x-y)]$$
$$=2\left[1-\left\{1-2\sin^2\left(\frac{x-y}{2}\right)\right\}\right] \qquad [\cos 2A=1-2\sin^2 A]$$
$$=4\sin^2\left(\frac{x-y}{2}\right)=\text{R.H.S.}$$

Question 5:

Prove that: $\sin x+\sin 3x+\sin 5x+\sin 7x=4\cos x\cos 2x\sin 4x$

Answer :

It is known that $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$.

∴L.H.S. = $\sin x+\sin 3x+\sin 5x+\sin 7x$

$$=(\sin x+\sin 5x)+(\sin 3x+\sin 7x)$$
$$=2\sin\left(\frac{x+5x}{2}\right)\cdot\cos\left(\frac{x-5x}{2}\right)+2\sin\left(\frac{3x+7x}{2}\right)\cos\left(\frac{3x-7x}{2}\right)$$
$$=2\sin 3x\cos(-2x)+2\sin 5x\ \cos(-2x)$$
$$=2\sin 3x\cos 2x+2\sin 5x\cos 2x$$
$$=2\cos 2x[\sin 3x+\sin 5x]$$
$$=2\cos 2x\left[2\sin\left(\frac{3x+5x}{2}\right)\cdot\cos\left(\frac{3x-5x}{2}\right)\right]$$
$$=2\cos 2x[2\sin 4x\cdot\cos(-x)]$$
$$=4\cos 2x\sin 4x\cos x\ =\ \text{R.H.S.}$$

Question 6:

Prove that: $\frac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}=\tan 6x$

Answer :

It is known that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right),\quad \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right).$$

L.H.S. = $\frac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}$

$$=\frac{\left[2\sin\left(\frac{7x+5x}{2}\right)\cdot\cos\left(\frac{7x-5x}{2}\right)\right]+\left[2\sin\left(\frac{9x+3x}{2}\right)\cdot\cos\left(\frac{9x-3x}{2}\right)\right]}{\left[2\cos\left(\frac{7x+5x}{2}\right)\cdot\cos\left(\frac{7x-5x}{2}\right)\right]+\left[2\cos\left(\frac{9x+3x}{2}\right)\cdot\cos\left(\frac{9x-3x}{2}\right)\right]}$$

$$=\frac{[2\sin 6x\cdot\cos x]+[2\sin 6x\cdot\cos 3x]}{[2\cos 6x\cdot\cos x]+[2\cos 6x\cdot\cos 3x]}$$

$$=\frac{2\sin 6x[\cos x+\cos 3x]}{2\cos 6x[\cos x+\cos 3x]}$$

= tan 6*x*

= R.H.S.

Question 7:

Prove that: $\sin 3x+\sin 2x-\sin x=4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$

Answer :

L.H.S. = $\sin 3x+\sin 2x-\sin x$

$$= \sin 3x + (\sin 2x - \sin x)$$

$$= \sin 3x + \left[2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)\right] \qquad \left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= \sin 3x + \left[2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)\right]$$

$$= \sin 3x + 2\cos\frac{3x}{2}\sin\frac{x}{2}$$

$$= 2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2} + 2\cos\frac{3x}{2}\sin\frac{x}{2} \qquad [\sin 2A = 2\sin A\cdot\cos B]$$

$$= 2\cos\left(\frac{3x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]$$

$$= 2\cos\left(\frac{3x}{2}\right)\left[2\sin\left\{\frac{\left(\frac{3x}{2}\right)+\left(\frac{x}{2}\right)}{2}\right\}\cos\left\{\frac{\left(\frac{3x}{2}\right)-\left(\frac{x}{2}\right)}{2}\right\}\right] \qquad \left[\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\cos\left(\frac{3x}{2}\right).2\sin x\cos\left(\frac{x}{2}\right)$$

$$= 4\sin x\cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) = \text{R.H.S.}$$

Question 8:

$\tan x = -\frac{4}{3}$, x in quadrant II

Answer :

Here, x is in quadrant II.

i.e., $\frac{\pi}{2} < x < \pi$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are all positive.

It is given that $\tan x = -\frac{4}{3}$.

$$\sec^2 x = 1 + \tan^2 x = 1 + \left(\frac{-4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\therefore \cos^2 x = \frac{9}{25}$$

$$\Rightarrow \cos x = \pm\frac{3}{5}$$

As x is in quadrant II, cos$x$ is negative.

$$\therefore \cos x = \frac{-3}{5}$$

$$\text{Now, } \cos x = 2\cos^2\frac{x}{2} - 1$$

$$\Rightarrow \frac{-3}{5} = 2\cos^2\frac{x}{2} - 1$$

$$\Rightarrow 2\cos^2\frac{x}{2} = 1 - \frac{3}{5}$$

$$\Rightarrow 2\cos^2\frac{x}{2} = \frac{2}{5}$$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{5}$$

$$\Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{5}} \qquad \left[\because \cos\frac{x}{2} \text{ is positive}\right]$$

$$\therefore \cos\frac{x}{2} = \frac{\sqrt{5}}{5}$$

$$\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1$$

$$\Rightarrow \sin^2\frac{x}{2} + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$$

$$\Rightarrow \sin^2\frac{x}{2} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow \sin\frac{x}{2} = \frac{2}{\sqrt{5}} \qquad \left[\because \ \sin\frac{x}{2} \text{ is positive}\right]$$

$$\therefore \sin\frac{x}{2} = \frac{2\sqrt{5}}{5}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)} = 2$$

Thus, the respective values of $\sin\frac{x}{2}, \ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{2\sqrt{5}}{5}, \ \frac{\sqrt{5}}{5}$, and 2.

Question 9:

Find $\sin\frac{x}{2}, \ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ for $\cos x = -\frac{1}{3}$, x in quadrant III

Answer :

Here, x is in quadrant III.

$$\text{i.e., } \pi < x < \frac{3\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Therefore, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are negative, whereas $\sin\frac{x}{2}$ is positive.

It is given that $\cos x = -\frac{1}{3}$.

$$\cos x = 1 - 2\sin^2 \frac{x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1-\left(-\frac{1}{3}\right)}{2} = \frac{\left(1+\frac{1}{3}\right)}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \qquad \left[\because \ \sin \frac{x}{2} \text{ is positive}\right]$$

$$\therefore \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Now, $\cos x = 2\cos^2 \frac{x}{2} - 1$

$$\Rightarrow \cos^2 \frac{x}{2} = \frac{1+\cos x}{2} = \frac{1+\left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} = \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3}$$

$$\Rightarrow \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \qquad \left[\because \ \cos \frac{x}{2} \text{ is negative}\right]$$

$$\therefore \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(\frac{-1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{\sqrt{6}}{3}$, $\frac{-\sqrt{3}}{3}$, and $-\sqrt{2}$.

Question 10:

Find $\sin\frac{x}{2},\ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ for $\sin x = \frac{1}{4}$, *x* in quadrant II

Answer :

Here, *x* is in quadrant II.

$$\text{i.e., } \frac{\pi}{2} < x < \pi$$
$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin\frac{x}{2}, \cos\frac{x}{2}$, and $\tan\frac{x}{2}$ are all positive.

$$\text{It is given that } \sin x = \frac{1}{4}.$$
$$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\Rightarrow \cos x = -\frac{\sqrt{15}}{4}$$ [cos*x* is negative in quadrant II]

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2} = \frac{1-\left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4+\sqrt{15}}{8}$$
$$\Rightarrow \sin\frac{x}{2} = \sqrt{\frac{4+\sqrt{15}}{8}} \qquad \left[\because \sin\frac{x}{2} \text{ is positive}\right]$$
$$= \sqrt{\frac{4+\sqrt{15}}{8} \times \frac{2}{2}}$$
$$= \sqrt{\frac{8+2\sqrt{15}}{16}}$$
$$= \frac{\sqrt{8+2\sqrt{15}}}{4}$$

$$\cos^2\frac{x}{2}=\frac{1+\cos x}{2}=\frac{1+\left(-\frac{\sqrt{15}}{4}\right)}{2}=\frac{4-\sqrt{15}}{8}$$

$$\Rightarrow \cos\frac{x}{2}=\sqrt{\frac{4-\sqrt{15}}{8}} \qquad \left[\because\ \cos\frac{x}{2}\ \text{is positive}\right]$$

$$=\sqrt{\frac{4-\sqrt{15}}{8}\times\frac{2}{2}}$$

$$=\sqrt{\frac{8-2\sqrt{15}}{16}}$$

$$=\frac{\sqrt{8-2\sqrt{15}}}{4}$$

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4}\right)}=\frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}}$$

$$=\sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}\times\frac{8+2\sqrt{15}}{8+2\sqrt{15}}}$$

$$=\sqrt{\frac{\left(8+2\sqrt{15}\right)^2}{64-60}}=\frac{8+2\sqrt{15}}{2}=4+\sqrt{15}$$

Thus, the respective values of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{\sqrt{8+2\sqrt{15}}}{4}$, $\frac{\sqrt{8-2\sqrt{15}}}{4}$,

and $4+\sqrt{15}$