Chapter –6 Application of derivatives

Miscellaneous

Question 1:

Using differentials, find the approximate value of each of the following.

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

Answer :

(a) Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$
$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\text{as } y = x^{\frac{1}{4}}\right)$$
$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4\times 8}\times\frac{1}{81} = \frac{1}{32\times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3}+0.010$ = 0.667 + 0.010

= 0.677.

(b) Consider $y=x^{-\frac{1}{5}}$. Let x = 32 and Δx = 1.

Then, $\Delta y=(x+\Delta x)^{-\frac{1}{5}}-x^{-\frac{1}{5}}=(33)^{-\frac{1}{5}}-(32)^{-\frac{1}{5}}=(33)^{-\frac{1}{5}}-\frac{1}{2}$

$$\therefore (33)^{-\frac{1}{5}}=\frac{1}{2}+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)(\Delta x)=\frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\text{as } y=x^{-\frac{1}{5}}\right)$$
$$=-\frac{1}{5(2)^{6}}(1)=-\frac{1}{320}=-0.003$$

Hence, the approximate value of $(33)^{-\frac{1}{5}}$ is $\frac{1}{2}+(-0.003)$

= 0.5 − 0.003 = 0.497.

Question 2:

Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum at $x = e$.

Answer :

The given function is $f(x)=\frac{\log x}{x}$.

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

Now, $f'(x) = 0$

$\Rightarrow 1 - \log x = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

$$\text{Now, } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1 - \log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$\text{Now, } f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, f is the maximum at $x = e$.

Question 3:

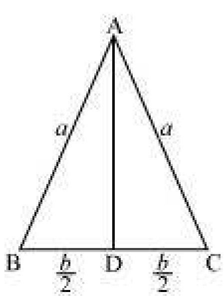
The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Answer :

Let ΔABC be isosceles where BC is the base of fixed length b.

Let the length of the two equal sides of ΔABC be a.

Draw AD$\perp$BC.



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$\text{AD} = \sqrt{a^2 - \frac{b^2}{4}}$$

$\therefore$ Area of triangle $(A) = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{1}{2}b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}}\frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}}\frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$\therefore \frac{da}{dt} = -3 \text{ cm/s}$

$$\therefore \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

Then, when $a = b$, we have:

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Hence, if the two equal sides are equal to the ba

Question 4:

Find the equation of the normal to curve $y^2 = 4x$ at the point (1, 2).

Answer :

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y\frac{dy}{dx}=4$$
$$\Rightarrow \frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$
$$\therefore \left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$$

Now, the slope of the normal at point (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}}=\frac{-1}{1}=-1.$

∴Equation of the normal at (1, 2) is $y - 2 = -1(x - 1)$.

$\Rightarrow y - 2 = -x + 1$

$\Rightarrow x + y - 3 = 0$

Question 5:

Show that the normal at any point θ to the curve

$x = a\cos\theta + a\theta\sin\theta,\ y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

Answer :

We have $x = a\cos\theta + a\theta\sin\theta$.

$$\therefore \frac{dx}{d\theta}=-a\sin\theta+a\sin\theta+a\theta\cos\theta=a\theta\cos\theta$$
$$y=a\sin\theta-a\theta\cos\theta$$
$$\therefore \frac{dy}{d\theta}=a\cos\theta-a\cos\theta+a\theta\sin\theta=a\theta\sin\theta$$
$$\therefore \frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

∴ Slope of the normal at any point θ is $-\frac{1}{\tan\theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}\left(x - a\cos\theta - a\theta\sin\theta\right)$$
$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$
$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$
$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|, \text{which is independent of } \theta.$$

Hence, the perpendicular distance of the normal from the origin is constant.

Question 6:

Find the intervals in which the function *f* given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing

Answer :

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^2 x-2\cos x+x\sin x\cos x+4\sin^2 x-2x\sin x-x\sin x\cos x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4\sin^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4-4\cos^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

Now, $f'(x)=0$

$\Rightarrow$ cos x = 0 or cos x = 4

But, cos $x \neq 4$

$\therefore$ cos x = 0

$$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divides (0, 2π) into three disjoint intervals i.e.,

$$\left(0,\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right), \text{and} \left(\frac{3\pi}{2},2\pi\right).$$

In intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$, $f'(x)>0.$

Thus, $f(x)$ is increasing for $0<x<\frac{x}{2}$ and $\frac{3\pi}{2}<x<2\pi.$

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), f'(x) < 0.$

Thus, $f(x)$ is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Question 7:

Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) increasing (ii) decreasing

Answer :

$$f(x) = x^3 + \frac{1}{x^3}$$
$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$
$$\text{Then, } f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x = \pm 1$$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals i.e., $(-\infty, -1), (-1, 1)$, and $(1, \infty)$.

In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0.$

Thus, when $x < -1$ and $x > 1$, f is increasing.

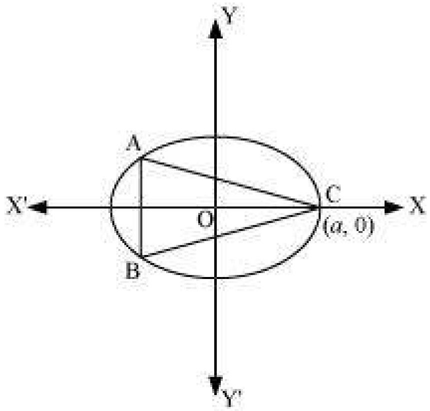
In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0.$

Thus, when $-1 < x < 1$, f is decreasing.

Question 8:

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Answer :



The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the major axis be along the x −axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at (a, 0).

Since the ellipse is symmetrical with respect to the x−axis and y −axis, we can assume the coordinates of A to be ($-x_1$, y_1) and the coordinates of B to be ($-x_1$, $-y_1$).

Now, we have $y_1=\pm\frac{b}{a}\sqrt{a^2-x_1^2}$.

∴Coordinates of A are $\left(-x_1,\ \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinates of B are $\left(x_1,\ -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (*A)* is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right|$$

$$\Rightarrow A=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2} \qquad ...(1)$$

$$\therefore \frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{a2\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+\left(a^2-x_1^2\right)-x_1^2\right]$$

$$=\frac{b\left(-2x_1^2-x_1a+a^2\right)}{a\sqrt{a^2-x_1^2}}$$

Now, $\frac{dA}{dx_1}=0$

$$\Rightarrow -2x_1^2-x_1a+a^2=0$$

$$\Rightarrow x_1=\frac{a\pm\sqrt{a^2-4(-2)\left(a^2\right)}}{2(-2)}$$

$$=\frac{a\pm\sqrt{9a^2}}{-4}$$

$$=\frac{a\pm 3a}{-4}$$

$$\Rightarrow x_1=-a,\ \frac{a}{2}$$

But, x_1 cannot be equal to a.

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{\sqrt{a^2 - x_1^2}\left(-4x_1 - a\right) - \left(-2x_1^2 - x_1 a + a^2\right)\dfrac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2}\right\}$$

$$= \frac{b}{a}\left\{\frac{\left(a^2 - x_1^2\right)\left(-4x_1 - a\right) + x_1\left(-2x_1^2 - x_1 a + a^2\right)}{\left(a^2 - x_1^2\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x^3 - 3a^2x - a^3}{\left(a^2 - x_1^2\right)^{\frac{3}{2}}}\right\}$$

Also, when $x_1 = \frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{2\dfrac{a^3}{8} - 3\dfrac{a^3}{2} - a^3}{\left(\dfrac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = \frac{b}{a}\left\{\frac{\dfrac{a^3}{4} - \dfrac{3}{2}a^3 - a^3}{\left(\dfrac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$= -\frac{b}{a}\left\{\frac{\dfrac{9}{4}a^3}{\left(\dfrac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} < 0$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}.$

∴ Maximum area of the triangle is given by,

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$
$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$
$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

Question 9:

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq meters for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

Answer :

Let l, b, and h represent the length, breadth, and height of the tank respectively.

Then, we have height $(h) = 2$ m

Volume of the tank = $8m^3$

Volume of the tank = $l \times b \times h$

$\therefore 8 = l \times b \times 2$

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

Now, area of the base = $lb = 4$

Area of the 4 walls $(A) = 2h\,(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$
$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$
$$\text{Now, } \frac{dA}{dl} = 0$$
$$\Rightarrow 1 - \frac{4}{l^2} = 0$$
$$\Rightarrow l^2 = 4$$
$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Therefore, we have $l = 4$.

$$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$$
$$\text{Now, } \frac{d^2 A}{dl^2} = \frac{32}{l^3}$$
$$\text{When } l = 2,\ \frac{d^2 A}{dl^2} = \frac{32}{8} = 4 > 0.$$

Thus, by second derivative test, the area is the minimum when $l = 2$.

We have $l = b = h = 2$.

∴Cost of building the base = Rs 70 × (lb) = Rs 70 (4) = Rs 280

Cost of building the walls = Rs $2h$ ($l + b$) × 45 = Rs 90 (2) (2 + 2)

= Rs 8 (90) = Rs 720

Required total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

Question 10:

The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Answer :

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$
$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (*A)* is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$
$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$
$$\text{Now}, \frac{dA}{dr} = 0$$
$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$
$$8r = k - 2\pi r$$
$$\Rightarrow (8 + 2\pi)r = k$$
$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$
$$\text{Now}, \ \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$
$$\therefore \text{When } r = \frac{k}{2(4+\pi)}, \ \frac{d^2A}{dr^2} > 0.$$
$$\therefore \text{The sum of the areas is least when } r = \frac{k}{2(4+\pi)}.$$
$$\text{When } r = \frac{k}{2(4+\pi)}, \ a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{k(4+\pi) - \pi k}{4(4+\pi)} = \frac{4k}{4(4+\pi)} = \frac{k}{4+\pi} = 2r.$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

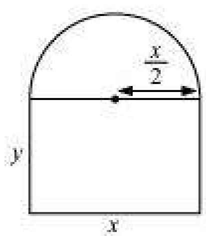
Question 11:

A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Answer :

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $=\frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore\ x+2y+\frac{\pi x}{2}=10$$

$$\Rightarrow x\left(1+\frac{\pi}{2}\right)+2y=10$$

$$\Rightarrow 2y=10-x\left(1+\frac{\pi}{2}\right)$$

$$\Rightarrow y=5-x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

∴Area of the window (*A)* is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

Thus, when $x = \frac{20}{\pi + 4}$ then $\frac{d^2A}{dx^2} < 0.$

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi + 4}$ m.

Now,

$$y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4} \text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by

$$\text{length} = \frac{20}{\pi+4} \text{ m and breadth} = \frac{10}{\pi+4} \text{ m.}$$

Question 12:

A point on the hypotenuse of a triangle is at distance *a* and *b* from the sides of the triangle.

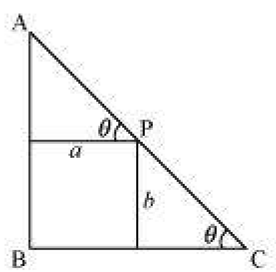
Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Answer :

Let ΔABC be right-angled at B. Let AB = *x* and BC = *y*.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of *a* and *b* from the sides AB and BC respectively.

Let ∠C = θ.



We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

PC = b cosec θ

And, AP = a sec θ

∴AC = AP + PC

⇒ AC = b cosec θ + a sec θ … (1)

$$\therefore \frac{d(\mathrm{AC})}{d\theta} = -b\operatorname{cosec}\theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(\mathrm{AC})}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \qquad \ldots(2)$$

It can be clearly shown that $\frac{d^2(\mathrm{AC})}{d\theta^2} < 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$\mathrm{AC} = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \qquad [\text{Using (1) and (2)}]$$

$$= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Question 13:

Find the points at which the function f given by $f(x)=(x-2)^4(x+1)^3$ has

(i) local maxima (ii) local minima

(ii) point of inflexion

Answer :

The given function is $f(x)=(x-2)^4(x+1)^3$.

$$\begin{aligned}\therefore f'(x) &= 4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4\\ &= (x-2)^3(x+1)^2\left[4(x+1)+3(x-2)\right]\\ &= (x-2)^3(x+1)^2(7x-2)\end{aligned}$$

$$\text{Now, } f'(x)=0 \Rightarrow x=-1 \text{ and } x=\frac{2}{7} \text{ or } x=2$$

Now, for values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}, f'(x)>0.$ Also, for values of x close to $\frac{2}{7}$ and to the right of $\frac{2}{7}, f'(x)<0.$

Thus, $x=\frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of $2, f'(x)<0.$ Also, for values of x close to 2 and to the right of 2, $f'(x)>0.$

Thus, $x = 2$ is the point of local minima.

Now, as the value of x varies through −1, $f'(x)$ does not changes its sign.

Thus, $x = -1$ is the point of inflexion.

Question 14:

Find the absolute maximum and minimum values of the function *f* given by

$f(x) = \cos^2 x + \sin x, x \in [0, \pi]$

Answer :

$$f(x) = \cos^2 x + \sin x$$
$$f'(x) = 2\cos x(-\sin x) + \cos x$$
$$= -2\sin x \cos x + \cos x$$
$$\text{Now, } f'(x) = 0$$
$$\Rightarrow 2\sin x \cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$$
$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \cos x = 0$$
$$\Rightarrow x = \frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x \in [0, \pi]$$

Now, evaluating the value of *f* at critical points $x = \frac{\pi}{2}$ and $x = \frac{\pi}{6}$ and at the end points of the interval $[0, \pi]$ (i.e., at $x = 0$ and $x = \pi$), we have:

$$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$
$$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$$
$$f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$$
$$f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$$

Hence, the absolute maximum value of *f* is $\frac{5}{4}$ occurring at $x = \frac{\pi}{6}$ and the absolute minimum value of *f* is 1 occurring at $x = 0, \frac{\pi}{2}, \text{and} \pi.$

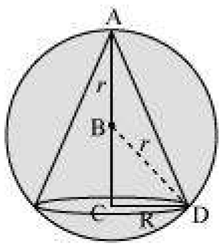
Question 15:

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Answer :

A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have:

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right) = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\therefore \frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2}{3}\pi R\sqrt{r^2 - R^2} + \frac{\pi R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2}{3}\pi R\sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi R\left(r^2 - R^2\right) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$\text{Now,} \frac{dV}{dR^2} = 0$$

$$\Rightarrow \frac{2\pi rR}{3} = \frac{3\pi R^3 - 2\pi Rr^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2\left(r^2 - R^2\right) = \left(3R^2 - 2r^2\right)^2$$

$$\Rightarrow 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$
$$\Rightarrow 9R^2 = 8r^2$$
$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\text{Now, } \frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2-R^2}\left(2\pi r^2 - 9\pi R^2\right) - \left(2\pi R r^2 - 3\pi R^3\right)(-6R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$
$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2-R^2}\left(2\pi r^2 - 9\pi R^2\right) + \left(2\pi R r^2 - 3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

∴ The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

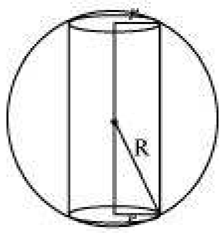
Question 17:

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Answer :

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume *(V)* of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2\sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r\left(R^2 - r^2\right) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\text{Now, } \frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$.

∴The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

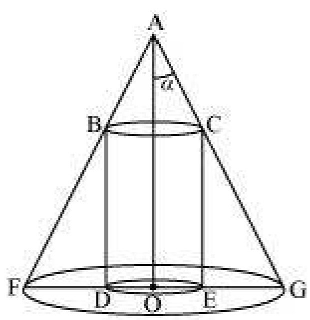
Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Question 18:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height *h* and semi vertical angle *α* is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3$ $\tan^2\alpha$.

Answer :

The given right circular cone of fixed height (*h*) and semi-vertical angle (*α*) can be drawn as:



Here, a cylinder of radius *R* and height *H* is inscribed in the cone.

Then, ∠GAO = *α*, OG = *r*, OA = *h*, OE = *R*, and CE = *H*.

We have,

$r = h \tan \alpha$

Now, since ΔAOG is similar to ΔCEG, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R} \qquad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h \tan \alpha}(h \tan \alpha - R) = \frac{1}{\tan \alpha}(h \tan \alpha - R)$$

Now, the volume (*V)* of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan \alpha}(h \tan \alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan \alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan \alpha}$$

$$\text{Now, } \frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi R h = \frac{3\pi R^2}{\tan \alpha}$$

$$\Rightarrow 2h \tan \alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3} \tan \alpha$$

$$\text{Now, } \frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan \alpha}$$

And, for $R = \frac{2h}{3} \tan \alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan \alpha}\left(\frac{2h}{3} \tan \alpha\right) = 2\pi h - 4\pi h = -2\pi h < 0$$

∴By second derivative test, the volume of the cylinder is the greatest when

$$R = \frac{2h}{3} \tan \alpha.$$

$$\text{When } R = \frac{2h}{3} \tan \alpha,\ H = \frac{1}{\tan \alpha}\left(h \tan \alpha - \frac{2h}{3} \tan \alpha\right) = \frac{1}{\tan \alpha}\left(\frac{h \tan \alpha}{3}\right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3\tan^2\alpha$$

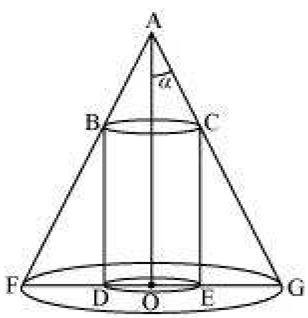
Hence, the given result is proved.

Question 18:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3$ $\tan^2\alpha$.

Answer :

The given right circular cone of fixed height (h) and semi-vertical angle (α) can be drawn as:



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$, and $CE = H$.

We have,

$r = h\tan\alpha$

Now, since ΔAOG is similar to ΔCEG, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r-R} \qquad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r-R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) = \frac{1}{\tan\alpha}(h\tan\alpha - R)$$

Now, the volume (*V)* of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi Rh - \frac{3\pi R^2}{\tan\alpha}$$

$$\text{Now}, \frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi Rh = \frac{3\pi R^2}{\tan\alpha}$$

$$\Rightarrow 2h\tan\alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3}\tan\alpha$$

$$\text{Now}, \frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan\alpha}$$

And, for $R = \frac{2h}{3}\tan\alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha}\left(\frac{2h}{3}\tan\alpha\right) = 2\pi h - 4\pi h = -2\pi h < 0$$

∴By second derivative test, the volume of the cylinder is the greatest when

$$R = \frac{2h}{3}\tan\alpha.$$

$$\text{When } R = \frac{2h}{3}\tan\alpha,\ H = \frac{1}{\tan\alpha}\left(h\tan\alpha - \frac{2h}{3}\tan\alpha\right) = \frac{1}{\tan\alpha}\left(\frac{h\tan\alpha}{3}\right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right)=\pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right)=\frac{4}{27}\pi h^3\tan^2\alpha$$

Hence, the given result is proved.

Question 20:

The slope of the tangent to the curve $x=t^2+3t-8, y=2t^2-2t-5$ at the point (2, −1) is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

Answer :

The given curve is $x=t^2+3t-8$ and $y=2t^2-2t-5.$

$$\therefore \frac{dx}{dt}=2t+3 \text{ and } \frac{dy}{dt}=4t-2$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{4t-2}{2t+3}$$

The given point is (2, −1).

At x = 2, we have:

$$\begin{aligned}
&t^2+3t-8=2\\
&\Rightarrow t^2+3t-10=0\\
&\Rightarrow (t-2)(t+5)=0\\
&\Rightarrow t=2 \text{ or } t=-5\\
&\text{At } y=-1, \text{ we have:}\\
&2t^2-2t-5=-1\\
&\Rightarrow 2t^2-2t-4=0\\
&\Rightarrow 2\left(t^2-t-2\right)=0\\
&\Rightarrow (t-2)(t+1)=0\\
&\Rightarrow t=2 \text{ or } t=-1
\end{aligned}$$

The common value of t is 2.

Hence, the slope of the tangent to the given curve at point (2, −1) is

$$\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}.$$

The correct answer is B.

Question 21:

The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

(A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

Answer :

The equation of the tangent to the given curve is $y = mx + 1$.

Now, substituting $y = mx + 1$ in $y^2 = 4x$, we get:

$$\Rightarrow (mx+1)^2 = 4x$$
$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$
$$\Rightarrow m^2x^2 + x(2m-4) + 1 = 0 \qquad \text{...(i)}$$

Since a tangent touches the curve at one point, the roots of equation (i) must be equal.

Therefore, we have:

$$\text{Discriminant} = 0$$
$$(2m-4)^2 - 4(m^2)(1) = 0$$
$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$
$$\Rightarrow 16 - 16m = 0$$
$$\Rightarrow m = 1$$

Hence, the required value of m is 1.

The correct answer is A.

Question 22:

The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ (B) $x - y = 0$

(C) $x + y + 1 = 0$ (D) $x - y = 1$

Answer :

The equation of the given curve is $2y + x^2 = 3$.

Differentiating with respect to x, we have:

$$\frac{2dy}{dx} + 2x = 0$$
$$\Rightarrow \frac{dy}{dx} = -x$$
$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}} = 1.$$

Hence, the equation of the normal to the given curve at (1, 1) is given as:

$$\Rightarrow y - 1 = 1(x - 1)$$
$$\Rightarrow y - 1 = x - 1$$
$$\Rightarrow x - y = 0$$

The correct answer is B.

Question 23:

The normal to the curve $x^2 = 4y$ passing (1, 2) is

(A) $x + y = 3$ (B) $x - y = 3$

(C) $x + y = 1$ (D) $x - y = 1$

Answer :

The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to x, we have:

$$2x = 4 \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h, k) is given by,

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

∴Equation of the normal at point (h, k) is given as:

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1, 2).

Therefore, we have:

$$2 - k = \frac{-2}{h}(1 - h) \text{ or } k = 2 + \frac{2}{h}(1 - h) \qquad \text{... (i)}$$

Since (h, k) lies on the curve $x^2 = 4y$, we have $h^2 = 4k$.

$$\Rightarrow k = \frac{h^2}{4}$$

From equation (i), we have:

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1-h)$$
$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$
$$\Rightarrow h^3 = 8$$
$$\Rightarrow h = 2$$
$$\therefore k = \frac{h^2}{4} \Rightarrow k = 1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y - 1 = \frac{-2}{2}(x-2)$$
$$\Rightarrow y - 1 = -(x-2)$$
$$\Rightarrow x + y = 3$$

The correct answer is A.

Question 24:

The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, \frac{-8}{3}\right)$

(C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$

Answer :

The equation of the given curve is $9y^2 = x^3$.

Differentiating with respect to x, we have:

$$9(2y)\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} = -\frac{6y_1}{x_1^2}.$$

∴ The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = \frac{-6y_1}{x_1^2}(x - x_1).$$
$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1 y_1$$
$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$
$$\Rightarrow \frac{6xy_1}{6x_1y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1 y_1 + x_1^2 y_1} = 1$$
$$\Rightarrow \frac{x}{\frac{x_1(6+x_1)}{6}} + \frac{y}{\frac{y_1(6+x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, We have:

$$\therefore \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$
$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$
$$\Rightarrow x_1^2 = 6y_1 \qquad \text{...(i)}$$

Also, the point (x_1, y_1) lies on the curve, so we have

$$9y_1^2 = x_1^3 \qquad \text{...(ii)}$$

From (i) and (ii), we have:

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

From (ii), we have:

$$9y_1^2 = (4)^3 = 64$$
$$\Rightarrow y_1^2 = \frac{64}{9}$$
$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$.

The correct answer is A.